Exercise Sheet 14

Exercise 1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes.

a) Recall the definition of all the data involved, and then define what the composition $g \circ f : X \to Z$ is. Verify that it satisfies all necessary properties.

b) For $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$ and $Z = \mathrm{Spec}(T)$ with f, g coming from ring homomorphisms $S \to R, T \to S$ the composition $g \circ f$ comes from the composite morphism $T \to S \to R$.

Note: Cynics might say that we should probably have done this in class, and they wouldn't be entirely wrong ...

Solution.

a) The underlying map of topological spaces for $g \circ f$ is indeed just the composition of the maps g, f. Clearly this is continuous as the composition of continuous maps. For any open set $U \subseteq Z$, the pullback map is defined as the composition

$$(g \circ f)^*_U : \mathcal{O}_Z(U) \xrightarrow{g^*_U} \mathcal{O}_Y(g^{-1}(U)) \xrightarrow{f^*_{g^{-1}(U)}} \mathcal{O}_X((g \circ f)^{-1}(U)) \,.$$

The fact that this is compatible under restrictions $U' \subseteq U \subseteq Z$ follows from the corresponding properties of the pullbacks f^*, g^*. The fact that for $p \in X$ the two maps $f^*_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ and $g^*_{f(p)} : \mathcal{O}_{Z,(g\circ f)(p)} \to \mathcal{O}_{Y,f(p)}$ are maps of local rings, implies the same for their composition $f^*_p \circ g^*_{f(p)} = (g \circ f)^*_p$.

b) As a map between affine schemes, the composition $g \circ f : \mathrm{Spec}(R) \to \mathrm{Spec}(T)$ is determined uniquely by its action on the global sections. By the above definition this is just

$$(g \circ f)^*_{\mathrm{Spec}(T)} : T \xrightarrow{g^*_{\mathrm{Spec}(T)}} S \xrightarrow{f^*_{\mathrm{Spec}(S)}} R$$

as claimed.

Exercise 2. Let $\mathcal{F}$ be an invertible sheaf (i.e. locally free of rank 1) on $\mathbb{A}^1_K = \mathrm{Spec} K[x]$ for K a field. Our goal below is to show that $\mathcal{F} \cong \mathcal{O}_{\mathbb{A}^1_K}$ is trivial.

a) Why is $\mathcal{F}$ is of the form $\mathcal{F} = \widetilde{M}$ for M a module over $K[x]$?

b) Let $D(f_i) \subseteq \mathbb{A}^1_K$ be a distinguished open such that there is an isomorphism

$$\varphi_i : \widetilde{R_{f_i}} = \mathcal{O}_{D(f_i)} \xrightarrow{\sim} \mathcal{F}|_{D(f_i)} = \widetilde{M_{f_i}}.$$

Let $\varphi_i(1) = m_i/f_i^{r_i}$ for $m_i \in M$ and $r_i \in \mathbb{N}$. Show that the map

$$s_i : \widetilde{R} \to \widetilde{M} \text{ induced by } R \to M, a \mapsto am_i$$

induces surjective maps of stalks $s_{i,p} : R_p \to M_p$ for all $p \in D(f_i)$.

c) Show that $\mathcal{F}$ is of the form $\mathcal{F} = \widetilde{M}$ for M a finitely generated module over $K[x]$.
Hint: Choose a finite cover of $\mathbb{A}^1_K$ by sets $D(f_1), \ldots, D(f_n)$ as in the previous part of the exercise and construct a surjection $R^n \to M$.

d) Prove that $\mathcal{F} \cong \mathcal{O}_{\mathbb{A}^1_K}$ is trivial.
Hint: Recall a certain statement about finitely generated modules over principal ideal domains.

Solution.

a) This follows from the fact that $\mathcal{F}$ is quasi-coherent. Indeed, every locally free sheaf is quasi-coherent, since there is an open cover by affine schemes $\mathrm{Spec} R_i$ for which it has the form $\widetilde{R_i^r}$. But then we had a result in class that any quasi-coherent sheaf on an affine scheme (like $\mathbb{A}^1_K = \mathrm{Spec} K[x]$) must come from a module over the corresponding ring (which is $K[x]$ in our example).

b) For $p \in D(f_i)$ the induced map of stalks is just

$$s_{i,p} : R_{f_i} \to M_{f_i}, \frac{a}{b} \mapsto \frac{am_i}{b}\,.$$

Comparing this to the map $\varphi_{i,p}$ of stalks above, we just see $s_{i,p} = f_i^{r_i} \cdot \varphi_{i,p}$. As $p \in D(f_i)$ we have that $f_i \in R \setminus p$ so f_i is a unit in R_p. As $s_{i,p}$ and $\varphi_{i,p}$ only differ by a unit, and $\varphi_{i,p}$ is an isomorphism, we have that $s_{i,p}$ is an isomorphism (and thus surjective).

c) As $\mathcal{F}$ is locally free, we can find a cover of $\mathbb{A}^1_K$ by distinguished opens $D(f_i)$ such that isomorphisms φ_i exist as in part b). We claim that finitely many of them already cover $\mathbb{A}^1_K$. To see this we can either use that $K[x]$ is Noetherian, and so $\mathrm{Spec} K[x]$ is also a Noetherian space. Or we can observe that the $D(f_i)$ covering $\mathbb{A}^1_K$ means that the intersection $V(f_i : i \in I)$ of their complements is empty. By the scheme-theoretic Nullstellensatz, this means that $\sqrt{\langle f_i : i \in I \rangle} = K[x]$ and so $1 = 1^m \in \langle f_i : i \in I \rangle$ which means that we find finitely many f_i and $t_i \in R$ with $1 = \sum_i t_i f_i$. From that equality one can see that $D(f_1) \cup \ldots \cup D(f_n) = \mathbb{A}^1_K$.

After choosing such a finite cover, consider the map

$$s : R^n \to M, (a_1, \ldots, a_n) \mapsto a_1 m_1 + \ldots + a_n m_n\,,$$

where the m_i are again as in part b). We claim that s is a surjective map of R-moduless (which would finish the proof). This surjectivity is equivalent to the exactness of the sequence $R^n \xrightarrow{s} M \to 0$. As we have seen in class, such an exactness can be checked at each prime $p \in \mathbb{A}^1_K$. But each p is contained in one of the $D(f_i)$, and by part b) we know that $s_{i,p}$ is then surjective. Since this is just one component of the map s_p it follows that s_p is surjective as well.

d) The relevant statement from the Hint is that any finitely generated module M over the PID $K[x]$ is a finite sum of cyclic modules, so that we have

$$M \cong (K[x]/\langle p_1 \rangle) \oplus \ldots \oplus (K[x]/\langle p_\ell \rangle)\,,$$

for $p_1, \ldots, p_\ell \in K[x]$, and we can assume that none of them is a unit since otherwise $K[x]/\langle p_i \rangle = \{0\}$. We claim that $\ell = 1$ and $p_1 = 0$, so that indeed $M \cong K[x]$ and $\mathcal{F} \cong \widetilde{K[x]} = \mathcal{O}_{\mathbb{A}^1}$.

To conclude the proof, we use that for all $q \in \operatorname{Spec}(K[x])$ we must have $M_q \cong \mathcal{F}_q \overset{!}{\cong} K[x]_q$ since $\mathcal{F}$ is locally free. Applying this first to $q = \langle 0 \rangle$ we get that

$$M_{\langle 0 \rangle} = (K(x)/\langle p_1 K(x) \rangle) \oplus \ldots \oplus (K(x)/\langle p_\ell K(x) \rangle) \overset{!}{\cong} K(x)\,,$$

where we use $K[x]_{\langle 0 \rangle} \cong K(x)$ and that localization commutes with quotients. For any polynomial $p \in K[x]$ we have

$$pK(x) = \begin{cases} 0 & \text{if } p = 0, \\ K(x) & \text{if } p \neq 0\,. \end{cases}$$

Thus the above equality can only happen if precisely one of the p_i (say p_1) is equal to zero, and all others are nonzero. It remains to show that $\ell = 1$. Otherwise, let q be any prime ideal containing p_2 (which exists since we assumed that p_2 is not a unit). Then we have $K[x]_q/\langle p_2 K[x]_q \rangle \neq \{0\}$ and so the equality

$$M_q = \underbrace{(K[x]_q/\langle 0 K[x]_q \rangle)}_{=K[x]_q} \oplus \underbrace{(K[x]_q/\langle p_2 K[x]_q \rangle)}_{\neq \{0\}} \oplus \ldots \cong K[x]_q$$

gives a contradiction.

Exercise 3. The goal of this exercise is to prove that for K an algebraically closed field, the set of automorphisms of $\mathbb{P}^n_K$ over K is isomorphic to the projective linear group $\mathrm{PGL}(n+1, K)$. The crucial input for the proof will be the following result, which you can use below:

Thm. Any invertible sheaf $\mathcal{L}$ on $\mathbb{P}^n_K$ is of the form $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^n_K}(d)$ for some $d \in \mathbb{Z}$.

a) Show that for $f : X \to Y$ a morphism and $\mathcal{F}, \mathcal{G}$ locally free sheaves on Y, one has $f^*(\mathcal{F} \otimes \mathcal{G}) \cong (f^*\mathcal{F}) \otimes (f^*\mathcal{G})$.
Hint: You can use without proof that it's sufficient to show this for X, Y affine schemes and $\mathcal{F}, \mathcal{G}$ trivial, i.e. direct sums of $\mathcal{O}_Y$.

b) Recall from class that any morphism $f : X \to \mathbb{P}^n_K$ is given by the data of an invertible sheaf $\mathcal{L}_f = f^*\mathcal{O}_{\mathbb{P}^n_K}(1)$ together with sections $s_0, \ldots, s_n \in \mathcal{L}(X)$ not vanishing simultaneously anywhere on X. Assume that $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ and $g : \mathbb{P}^n_K \to \mathbb{P}^n_K$ are given by line bundles $\mathcal{L}_f = \mathcal{O}_{\mathbb{P}^n_K}(d_f)$ and $\mathcal{L}_g = \mathcal{O}_{\mathbb{P}^n_K}(d_g)$. Show that we have $d_f, d_g \geq 0$ and the composition $g \circ f$ is given by $\mathcal{L}_{g \circ f} \cong \mathcal{O}_{\mathbb{P}^n_K}(d_f \cdot d_g)$.

c) Conclude that for any isomorphism $f : \mathbb{P}^n_K \xrightarrow{\sim} \mathbb{P}^n_K$ one has $f^*\mathcal{O}_{\mathbb{P}^n_K}(1) \cong \mathcal{O}_{\mathbb{P}^n_K}(1)$.

d) Finish the proof that any automorphism $f \in \operatorname{Aut}_K(\mathbb{P}^n_K, \mathbb{P}^n_K)$ is given by a projective linear map in $\mathrm{PGL}(n+1, K)$.

Solution.

a) Let's first prove the statement in the suggested setting of $X = \operatorname{Spec}(R)$ and $Y = \operatorname{Spec}(S)$ with $\mathcal{F} = \widetilde{S^a}$ and $\mathcal{G} = \widetilde{S^b}$. Then we use that pullbacks and tensor products are easy for quasi-coherent sheaves on affine schemes:

$$f^*(\mathcal{F} \otimes \mathcal{G}) = f^*(\underbrace{\widetilde{S^a} \otimes \widetilde{S^b}}_{=\widetilde{S^a \otimes_S S^b} \cong \widetilde{S^{ab}}}) \cong (\widetilde{S^{ab} \otimes_S R}) = \widetilde{R^{ab}} \cong \widetilde{R^a} \otimes \widetilde{R^b} \cong (f^*\mathcal{F}) \otimes (f^*\mathcal{G})$$

Using some gluing argument, one can generalize the proof to arbitrary X, Y.

b) Using that $\mathcal{O}_{\mathbb{P}^n_K}(a) \otimes \mathcal{O}_{\mathbb{P}^n_K}(b) \overset{(*)}{\cong} \mathcal{O}_{\mathbb{P}^n_K}(a+b)$, we have

$$\mathcal{L}_{g\circ f} = (g \circ f)^* \mathcal{O}_{\mathbb{P}^n_K}(1) = f^* g^* \mathcal{O}_{\mathbb{P}^n_K}(1) = f^* \mathcal{O}_{\mathbb{P}^n_K}(d_g) \overset{(*)}{=} f^* \underbrace{\mathcal{O}_{\mathbb{P}^n_K}(1) \otimes \ldots \otimes \mathcal{O}_{\mathbb{P}^n_K}(1)}_{d_g \text{ times}}$$

$$\overset{\text{a)}}{\cong} \underbrace{f^*\mathcal{O}_{\mathbb{P}^n_K}(1) \otimes \ldots \otimes f^*\mathcal{O}_{\mathbb{P}^n_K}(1)}_{d_g \text{ times}} \cong \underbrace{\mathcal{O}_{\mathbb{P}^n_K}(d_f) \otimes \ldots \otimes \mathcal{O}_{\mathbb{P}^n_K}(d_f)}_{d_g \text{ times}} \overset{(*)}{\cong} \mathcal{O}_{\mathbb{P}^n_K}(d_f \cdot d_g)\,.$$

For the inequalities $d_f, d_g \geq 0$: if say $d_f < 0$, then $\mathcal{L}_f(\mathbb{P}^n_K) = \{0\}$ so the sections $s_0, \ldots, s_n \in \mathcal{L}_f(\mathbb{P}^n_K)$ defining f would all vanish, giving a contradiction to them not all vanishing simultaneously.

c) Let g be the inverse of f and d_f, d_g as in part b), then

$$\mathcal{L}_{g\circ f} = (g \circ f)^* \mathcal{O}_{\mathbb{P}^n_K}(1) = (\mathrm{id}_{\mathbb{P}^n_K})^* \mathcal{O}_{\mathbb{P}^n_K}(1) = \mathcal{O}_{\mathbb{P}^n_K}(1).$$

But by part b) this is also isomorphic to $\mathcal{O}_{\mathbb{P}^n_K}(d_f d_g)$, so $d_f d_g = 1$. Since also $d_f, d_g \geq 0$ this forces $d_f = d_g = 1$ and so $f^*\mathcal{O}_{\mathbb{P}^n_K}(1) \cong \mathcal{O}_{\mathbb{P}^n_K}(1)$.

d) By part c) and the classification theorem for maps to projective space, the morphism f is given by sections

$$s_0, \ldots, s_n \in \mathcal{O}_{\mathbb{P}^n_K}(1)(\mathbb{P}^n_K) = \mathrm{Lin}_K(x_0, \ldots, x_n)\,,$$

in other words by $n+1$ homogeneous linear polynomials in the coordinates $x_0, \ldots, x_n$. This shows that f is just the multiplication by an $(n+1) \times (n+1)$ matrix over K (containing the coefficients of these polynomials). If that matrix had a non-trivial element x in its kernel, then $s_0(x) = \ldots = s_n(x) = 0$, giving a contradiction to the sections not vanishing simultaneously at any point. Thus the matrix is invertible, and hence f is a projective linear automorphism as claimed.