Exercise Sheet 2

Exercise 1. Let $X \subset \mathbb{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X}$.

Solution. From the definitions it's clear that $X \subseteq V(I(X))$ and since that latter set is closed, also $\overline{X}$ is contained in it. Conversely, assume that Y is a closed set containing X, then clearly $I(Y) \subseteq I(X)$ and hence $V(I(X)) \subseteq V(I(Y)) = Y$, where we use that $V(-)$ and $I(-)$ are inclusion-reversing bijections between the set of affine varieties and the set of radical ideals in $K[x_1, \ldots, x_n]$. Applying this to $Y = \overline{X}$ yields the desired inclusion.

Exercise 2. (Topology marathon) Let X be a topological space. Show the following statements.

a) If $X = X_1 \cup \ldots \cup X_n$ with X_i Noetherian, then X is Noetherian as well.

b) If $Y \subseteq A \subseteq X$ is closed in the subspace topology of A, then $\overline{Y} \cap A = Y$.

c) A set $A \subseteq X$ is irreducible if and only if $\overline{A}$ is irreducible.

d) If X is irreducible and $U \subseteq X$ is open, then U is irreducible.

e) If $f : X \to Y$ is continuous and X is connected, then so is $f(X)$.

f) If $f : X \to Y$ is continuous and X is irreducible, then so is $f(X)$.

Solution.

a) Assume we had a decreasing chain $Y_0 \supseteq Y_1 \supseteq \ldots$ of closed subsets of X. Intersecting with the X_i would yield decreasing chains of closed subsets of X_i. Since X_i is Noetherian, they contain only finitely many strict inclusions, so there exists N after which the sequences stabilize. But then also the sequence

$$Y_m = Y_m \cap (X_1 \cup \ldots \cup X_n) = (Y_m \cap X_1) \cup \ldots \cup (Y_m \cap X_n)$$

stabilizes for $m \geq N$, showing that X is Noetherian.

b) The inclusion $Y \subseteq \overline{Y} \cap A$ is clear. Conversely, the set Y is closed in A if there exists a closed set $Z \subseteq X$ with $Y = Z \cap A$. Since $\overline{Y}$ is the smallest set containing Y, we have $\overline{Y} \subseteq Z$ and thus $\overline{Y} \cap A \subseteq Z \cap A = Y$.

c) Assume that A is irreducible and let $\overline{A} = B_1 \cup B_2$ be a cover by closed sets. Then $A = (A \cap B_1) \cup (A \cap B_2)$ is a cover of A by closed sets, forcing e.g. $A \cap B_1 = A$, i.e. $A \subseteq B_1$. Taking closure on both sides gives $\overline{A} \subseteq \overline{B}_1 = B_1$ and thus $\overline{A} = B_1$.

Conversely, assume that $\overline{A}$ is irreducible and let $A = A_1 \cup A_2$ be a cover by closed sets. Taking closure on both sides we have $\overline{A} = \overline{A}_1 \cup \overline{A}_2$ forcing e.g. $\overline{A} = \overline{A}_1$. But then by part b) we have $A = \overline{A}_1 \cap A = A_1$.

d) If $U = \emptyset$ then it is irreducible from the definition. On the other hand, for $U \neq \emptyset$ we saw in the lecture that U is dense, so $\overline{U} = X$. Since this closure is irreducible, by part c) also U itself is irreducible.

e) Assume $f(X) = Z_1 \cup Z_2$ with Z_1, Z_2 closed and disjoint. Then $X = f^{-1}(Z_1) \cup f^{-1}(Z_2)$ is a closed disjoint cover of X. Since X is connected, this forces e.g. $f^{-1}(Z_1) = X$, but then applying f on both sides (and using that $f : X \to f(X)$ is surjective) we have $Z_1 = f(X)$. This proves that $f(X)$ is connected.

f) Repeating the argument from part e) removing the word "disjoint" proves that $f(X)$ is irreducible.

Exercise 3. Calculate the irreducible components of $X = V(J)$ for

$$J = \langle y^2 - x^4, x^2 - 2x^3 - x^2y + 2xy + y^2 - y \rangle \subseteq K[x, y].$$

Hint: The answer depends on the characteristic of K.

Solution. Factoring the generators we have

$$J = \langle (y - x^2)(y + x^2), (y - x^2)(2x + y - 1) \rangle = \underbrace{\langle y - x^2 \rangle}_{=J_1} \cdot \underbrace{\langle y + x^2, 2x + y - 1 \rangle}_{=J_2}.$$

As a result, we have $X = V(J_1) \cup V(J_2) =: X_1 \cup X_2$. Since $K[x, y]/J_1 \cong K[x]$ is a domain, we have that J_1 is prime and X_1 is irreducible. On the other hand we have

$$X_2 = V(y+x^2, 2x+y-1) = V(-2x+1+x^2, 2x+y-1) = V(x-1, 2x+y-1) = \{(1, -1)\}.$$

Hence X_2 is irreducible as well. For $\mathrm{char}(K) \neq 2$ we have that $(1, -1) \notin X_1$ and so $X = X_1 \sqcup X_2$ is the irreducible decomposition. For $\mathrm{char}(K) = 2$ we do have $X_2 \subseteq X_1$ and so $X = X_1$ is the decomposition.

Exercise 4. Let $\{U_i : i \in I\}$ be an open cover of a topological space X and assume that $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. Show:

a) If U_i is connected for all i, then so is X.

b) If U_i is irreducible for all i, then so is X.

Solution.

a) Assume that $X = X_1 \cup X_2$ is a disjoint closed cover of X, then $U_i = (X_1 \cap U_i) \cup (X_2 \cap U_i)$ is a disjoint closed cover of U_i. Since U_i is connected, this means that one of these two sets must be all of U_i, so there exists $k_i \in \{1, 2\}$ with $U_i \subseteq X_{k_i}$. We claim that the k_i must either all be 1 or all be 2. Assume otherwise, i.e. $U_{i_1} \subseteq X_1$ and $U_{i_2} \subseteq X_2$. This gives a contradiction since $\emptyset \neq U_{i_1} \cap U_{i_2} \subseteq X_1 \cap X_2 = \emptyset$.

b) For any i, j the non-empty open set $U_i \cap U_j$ is dense in both U_i and U_j, since these two sets are irreducible and hence *any* non-empty open subset of them is dense. Picking any $i_0 \in I$ we then claim that $\overline{U}_{i_0} = X$ which finishes the proof since X is then irreducible as the closure of an irreducible set. The claim just follows from the fact that for all $j \in I$ we have $\overline{U}_{i_0} \supseteq \overline{U_{i_0} \cap U_j} \supseteq U_j$ where the second containment was shown in the first sentence of the current proof. Taking the union over all j proves $\overline{U}_{i_0} \supseteq \bigcup_j U_j = X$.

Exercise 5. Let X be a topological space. Prove:

a) If $\{U_i : i \in I\}$ is an open cover of X, then $\dim X = \sup\{\dim U_i : i \in I\}$.

b) If X is an irreducible affine variety and $U \subseteq X$ a non-empty open subset, then $\dim X = \dim U$.

c) Does the statement from (b) hold more generally for any irreducible topological space?

Solution.

a) Since $U_i \subseteq X$ we have $\dim U_i \leq \dim X$, which shows the inequality "$\geq$". For the other inequality, assume we have a strict ascending chain $X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n \subseteq X$ of closed irreducible subsets of X. Choose i_0 such that $X_0 \cap U_{i_0} \neq \emptyset$. Then we claim that

$$X_0 \cap U_{i_0} \subseteq X_1 \cap U_{i_0} \subseteq \ldots \subseteq X_n \cap U_{i_0} \subseteq U_{i_0} \tag{1}$$

is a *strict* ascending chain. Once we prove this claim, we obtain $n \leq \dim U_{i_0}$ and taking the supremum over all n as above, we see that $\dim X \leq \sup\{\dim U_i : i \in I\}$ as desired.

To show the claim, assume that $X_k \cap U_{i_0} = X_{k+1} \cap U_{i_0}$. But then $X_{k+1} = X_k \cup (X_{k+1} \setminus U_{i_0})$ is a cover of the irreducible set X_{k+1} by two closed subsets. However $X_k \subsetneq X_{k+1}$ is a strict containment, and similarly $X_{k+1} \setminus U_{i_0} \subsetneq X_{k+1}$ since $\emptyset \neq X_0 \cap U_{i_0} \subseteq X_{k+1} \cap U_{i_0}$ is non-empty. This gives a contradiction, and proves the claim.

b) We prove the statement by induction on $\dim X$. For $\dim X = 0$ we necessarily have $X = \{a\}$ which forces $U = X$ since U is non-empty, and proves $\dim X = \dim U$.

For the induction step, assume that $X \subseteq \mathbb{A}^m$ is of dimension $n+1$. First, we remark that by Exercise 2 d) the set U is irreducible itself, and dense in X. Moreover, it must be of dimension at least 1 (since otherwise U is a single point, and so its closure X is still a single point, hence of dimension 0, giving a contradiction).

Choose $a \in U \subseteq X$ and let $f \in K[x_1, \ldots, x_m]$ be a linear polynomial vanishing at a but not constant on U (which is possible since U being of positive dimension, it must have at least one other point apart from a). Then $0 \neq f \in A(X)$ and by Krull's principal ideal theorem, any irreducible component X' of $V_X(f)$ has dimension n. Choose such a component X' containing $a \in V_X(f)$. Then X' is an irreducible affine variety of dimension n and $U \cap X'$ is a non-empty open subset, hence by the induction hypothesis we have $n = \dim X' = \dim U \cap X'$. Let $V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_n \subseteq U \cap X'$ be a strict ascending chain of closed irreducible subsets of $U \cap X'$. Then it can be extended by the strict inclusion $V_n \subsetneq U$ to a chain of length $n+1$, showing $\dim U \geq n+1 = \dim X \geq \dim U$ proving the desired equality. The inclusion of V_n is strict in U since

$$V_n \subseteq U \cap X' \subseteq U \cap V_X(f) \subsetneq U$$

as by assumption f is not constant (and hence not identically equal to zero) on U.

c) No: take $X = \{0, 1\}$ with the topology having closed sets $\mathcal{C} = \{\emptyset, \{1\}, \{1, 2\}\}$. All these closed sets are irreducible (since none is a union of two strictly contained smaller closed sets), and in particular $\{1\} \subseteq \{1, 2\} = X$ is the unique maximal chain of irreducible non-empty subsets of X, showing $\dim X = 1$. However, the open set $U = \{2\} \subseteq X$ is non-empty, but has $\dim U = 0$.