Exercise Sheet 3

Exercise 1. Let $\varphi, \psi \in \mathcal{F}(U)$ be two sections of a sheaf $\mathcal{F}$ on an open subset U of a topological space X. Show:

a) If φ and ψ agree in all stalks i.e., $[(U, \varphi)] = [(U, \psi)] \in \mathcal{F}_a$ for all $a \in U$ then $\varphi = \psi$.

b) If $\mathcal{F} = \mathcal{O}_X$ is the sheaf of regular functions on an irreducible affine variety X then we can already conclude that $\varphi = \psi$ if we only know that they agree in *one* stalk $\mathcal{F}_a$ for $a \in U$.
Hint: It might help to first cover U by distinguished affine open subsets.

c) For a general sheaf $\mathcal{F}$ on a topological space X the statement of (b) is false.

Solution.

a) Given $a \in U$, the equality $[(U, \varphi)] = [(U, \psi)] \in \mathcal{F}_a$ by definition means that there is an open set $a \in U_a \subseteq U$ with $\varphi|_{U_a} = \psi|_{U_a}$. Thus on the open cover $\{U_a : a \in U\}$ of U we have the system of sections $\psi|_{U_a} \in \mathcal{F}(U_a)$ which of course are compatible on the overlaps $U_a \cap U_b$. By the sheaf axiom for $\mathcal{F}$ there is a unique section in $\mathcal{F}(U)$ restricting to the $\psi|_{U_a} \in \mathcal{F}(U_a)$. But both φ and ψ have this property, and thus by uniqueness they must be equal.

b) First, we can cover U by distinguished open sets $D(f)$ and as above it suffices to show that the restrictions of φ, ψ agree on $D(f)$. So without loss of generality let us assume that $U = D(f)$ to start with, in which case $\varphi, \psi \in \mathcal{O}_X(U) = A(X)_f$ can be seen as elements of the localization of $A(X)$ at f. We have seen that for X an affine irreducible variety, its coordinate ring $A(X)$ is a domain. The equality $[\varphi] = [\psi] \in \mathcal{O}_{X,a} = A(X)_{I_X(a)}$ then implies that the images of φ, ψ under the localization map $A(X)_f \to A(X)_{I_X(a)}$ are equal. But $A(X)$ being a domain implies that this localization is injective, so $\varphi = \psi$ as desired.

c) Choose $X = U = \{0, 1\}$ with the discrete topology and $\mathcal{F}$ the sheaf of (arbitrary) functions to $\mathbb{R}$. Then $\varphi(x) = 0$ and $\psi(x) = x$ agree in

$$\mathcal{F}_0 = \left(\underbrace{\mathcal{F}(\{0,1\})}_{\cong \mathbb{R}^2} \cup \underbrace{\mathcal{F}(\{0\})}_{\cong \mathbb{R}} \right) / \sim \cong \quad \mathbb{R}$$

but do not agree at 1. Here in the computation of $\mathcal{F}_0$ a function $(0 \mapsto a, 1 \mapsto b) \in \mathcal{F}(\{0, 1\})$ is equivalent to $(0 \mapsto c) \in \mathcal{F}(\{0\})$ if $c = a$. Thus every equivalence class has a unique representative in $\mathcal{F}(\{0\}) = \mathbb{R}$ and thus the set $\mathcal{F}_0$ of equivalence classes is exactly isomorphic to $\mathbb{R}$ (via evaluation at 0).

Exercise 2. Let a be any point on the real line $\mathbb{R}$. For which of the following sheaves $\mathcal{F}$ on $\mathbb{R}$ (with the standard topology) is the stalk $\mathcal{F}_a$ actually a local ring in the algebraic sense (i.e., it has exactly one maximal ideal)?

a) $\mathcal{F}$ is the sheaf of continuous functions;

b) $\mathcal{F}$ is the sheaf of locally polynomial functions.

Solution.

a) We claim that the unique maximal ideal of $\mathcal{F}_a$ is given by

$$\mathfrak{m}_a = \{f \in \mathcal{F}_a : f(a) = 0\} = \ker(\mathrm{ev}_a : \mathcal{F}_a \to \mathbb{R}) .$$

Since ev_a is surjective (a preimage of $b \in \mathbb{R}$ is the constant function $x \mapsto b$), we have $\mathbb{R} = \mathcal{F}_a/\mathfrak{m}_a$ by the second isomorphism theorem. Since this is a field, we have that $\mathfrak{m}_a$ is maximal. On the other hand, let $[(U, g)] \in \mathcal{F}_a \setminus \mathfrak{m}_a$, then we claim that this element is a unit in $\mathcal{F}_a$. Indeed, since $g(a) \neq 0$ we have $V = \{x \in U : g(x) \neq 0\}$ is an open neighborhood of a. Then $[(U, g)] \cdot [(V, 1/g)] = [(V, 1)] = 1 \in \mathcal{F}_a$, showing $[(U, g)]$ is a unit. But then the complement of $\mathfrak{m}_a$ is the set of units in $\mathcal{F}_a$, which proves that $\mathfrak{m}_a$ is the unique maximal ideal (any other maximal ideal would have to contain at least one element outside $\mathfrak{m}_a$, which would force the ideal to be all of $\mathcal{F}_a$, a contradiction).

b) By the first part of the argument above, the ideal $\mathfrak{m}_a = \ker(\mathrm{ev}_a)$ is maximal. However, it is not the only maximal ideal: consider the function $f = x - a + 1 \in \mathcal{F}_a$. We claim that f is not a unit. Indeed, otherwise its inverse g would have to be given by a polynomial in some open neighborhood U of a. But then the equality $f \cdot g = 1$ on U would force this equality to be true in $\mathbb{R}[x]$ as well (since U is infinite). Since f has degree 1, this gives a contradiction.

As a result, since f is not a unit, the ideal $\langle f \rangle$ generated by f is not all of $\mathcal{F}_a$. But then it is contained in some maximal ideal $\mathfrak{m}'$ of $\mathcal{F}_a$. Since $f \notin \mathfrak{m}_a$ (as $f(a) = 1 \neq 0$), we have $\mathfrak{m}' \neq \mathfrak{m}_a$, giving a second maximal ideal in $\mathcal{F}_a$.

Exercise 3. Let Y be a non-empty irreducible subvariety of an equidimensional affine variety X and set $U = X \setminus Y$.

a) Assume that $A(X)$ is a unique factorization domain. Show that $\mathcal{O}_X(U) = A(X)$ if and only if $\mathrm{codim}_X Y \geq 2$.

b) Show by example that the equivalence of (a) is in general false if $A(X)$ is not assumed to be a unique factorization domain.
Note: It's pretty hard (but not impossible) to give an example with X irreducible, but feel free to look for a reducible example.

Solution.

a) Let $Y = V_X(f_1, \ldots, f_r)$ for $f_1, \ldots, f_r \in A(X)$ nonzero, then $U = D(f_1) \cup \ldots \cup D(f_r)$ is a cover by distinguished open subsets. Moreover, the fact that $A(X)$ is a unique factorization domain (and thus in particular an integral domain) implies that X is irreducible.

Now first assume that $\operatorname{codim}_X(Y) \leq 1$. For $\operatorname{codim}_X(Y) = 0$ the inclusion $Y \subseteq X$ with X, Y irreducible already implies $Y = X$. Thus $U = \emptyset$ and $\mathcal{O}_X(U) = \{0\}$ which is not a domain and hence not equal to $A(X)$. On the other hand, for $\operatorname{codim}_X Y = 1$ we have seen in the lecture that $I_X(Y) = \langle f \rangle$ is principal. But then $\mathcal{O}_X(U) = \mathcal{O}_X(D(f)) = A(X)_f \neq A(X)$, where we use that f is not a unit since $Y \neq \emptyset$. This proves one direction of the claimed equivalence.

On the other hand, assume $\operatorname{codim}_X Y \geq 2$, then in the representation $Y = V_X(f_1, \ldots, f_r)$ we necessarily have that in the irreducible decompositions of the f_i in the UFD $A(X)$, not all f_i share one common irreducible factor g. Otherwise, we would have $V_X(g) \subseteq Y$, which is a contradiction to the codimension of Y being at least two (since by Zariski's principal ideal theorem, the codimension of $V(g)$ would be at most 1). Then assume that $\varphi \in \mathcal{O}_X(U)$, then we want to show that $\varphi \in A(X)$ is a restriction of a function from all of X. We begin by noting that there is a representation

$$\varphi|_{D(f_i)} = \frac{g_i}{f_i^{m_i}} \in \mathcal{O}_X(D(f_i)) = A(X)_{f_i}\,.$$

If one of the $m_i = 0$, then the equality $\varphi = g_i$ on the dense open subset $D(f_i) \subseteq X$ implies $\varphi = g_i \in A(X)$ on all of X, finishing the proof. Otherwise, we can assume that f_i does not divide g_i. For the irreducible decomposition

$$f_i = h_{i,1}^{e_{i,1}} \cdots h_{i,r_i}^{e_{i,r_i}}$$

there must thus be an index k_i such that $h_{i,k_i}^{e_{i,k_i}}$ does not divide g_i.

Since X is irreducible, the non-empty open sets $D(f_i)$ and $D(f_j)$ intersect, and the equality $g_i/f_i^{m_i} = g_j/f_j^{m_j}$ in $\mathcal{O}_X(D(f_i f_j)) = A(X)_{f_i f_j}$ implies $g_i f_j^{m_j} = g_j f_i^{m_i} \in A(X)$. Looking at powers of the irreducible factor h_{i,k_i} on both sides (and using $m_i \geq 1$), we see that h_{i,k_i} must divide $f_j^{m_j}$ and thus f_j for all j. But this gives a contradiction to our assumption that not all the functions $f_1, \ldots, f_r$ share a common irreducible factor in $A(X)$.

b) Take $X = \mathbb{A}^2 \times \{(0,0)\} \cup \{(0,0)\} \times \mathbb{A}^2 \subseteq \mathbb{A}^4$ and $Y = \{(0,0,0,0)\}$. Then

$$U = X \setminus Y = \underbrace{(\mathbb{A}^2 \setminus \{(0,0)\}) \times \{(0,0)\}}_{=:U_1} \sqcup \underbrace{\{(0,0)\} \times (\mathbb{A}^2 \setminus \{(0,0)\})}_{=:U_2}\,.$$

Then U_1, U_2 are open in U and thus the function $\varphi \in \mathcal{O}_X(U)$ which is 0 on U_1 and 1 on U_2 is regular (as can be checked on the open cover $U = U_1 \cup U_2$). However, it cannot have an extension to $X = U \cup \{(0,0,0,0)\}$ because then $(0,0,0,0)$ would be both in the closure of $V_X(\varphi)$ and $V_X(\varphi - 1)$, which are necessarily disjoint sets.

Take $X = V(a^2c - b^2, a^3d - b^3, c^3 - d^2) \subseteq \mathbb{A}^4$. The function $\frac{d}{c}$ is a section of the structure sheaf over $D(c) \subset X$ and the function $\frac{b}{a}$ is a section of the structure sheaf over $D(a) \subset X$. These sections coincide over $D(ac) \subset X$, therefore they define a regular function on $D(c) \cup D(a) \subset X$.

But $X \setminus (D(a) \cup D(c)) = V(a,c) \cap X = (0,0,0,0)$ so removing the point $(0,0,0,0)$ from X yields a regular function defined by $\frac{d}{c}$ and $\frac{b}{a}$. It remains to see that there is no regular function extending these to X.

Indeed, one can check (Exercise!) that the map $\phi\colon \mathbb{C}[a,b,c,d] \to \mathbb{C}[x,y]$ given by

$$a \mapsto x, b \mapsto xy, c \mapsto y^2, d \mapsto y^3$$

has kernel $(a^2c - b^2, a^3d - b^3, c^3 - d^2)$ and hence induces an isomorphism between the algebra $\mathbb{C}[a, b, c, d]/(a^2c - b^2, a^3d - b^3, c^3 - d^2)$ and a subalgebra R of $\mathbb{C}[x, y]$ generated by x, xy, y^2, y^3. As the latter is an integral domain, this in particular implies that $R \simeq A(X)$ and if the function $\frac{d}{c}$ were extendable to a regular function on X its image in $\mathbb{C}[x, y]$ would be y which is not in R, so we get a contradiction.

Exercise 4. Let $\mathcal{F}$ be a sheaf on a topological space X and let Y be a non-empty irreducible closed subset of X. We define the *stalk of* $\mathcal{F}$ *at* Y to be

$$\mathcal{F}_Y := \{(U, \varphi) : U \text{ is an open subset of } X \text{ with } U \cap Y \neq \emptyset \text{ and } \varphi \in \mathcal{F}(U)\}/\sim$$

where $(U, \varphi) \sim (U', \varphi')$ if and only if there is an open set $V \subset U \cap U'$ with $V \cap Y \neq \emptyset$ and $\varphi|_V = \varphi'|_V$. It therefore describes functions in an arbitrarily small neighborhood of an arbitrary dense open subset of Y.

If Y is a non-empty irreducible subvariety of an affine variety X prove that the stalk $\mathcal{O}_{X,Y}$ of $\mathcal{O}_X$ at Y is a K-algebra isomorphic to the localization $A(X)_{I(Y)}$ (hence giving a geometric meaning to this algebraic localization).

Solution. Consider the map

$$\Psi : A(X)_{I(Y)} \to \mathcal{O}_{X,Y}, \frac{f}{g} \mapsto [(D(g), x \mapsto \frac{f(x)}{g(x)})]$$

First note that $g \notin I(Y)$ implies $D(g) \cap Y \neq \emptyset$, so the map is well-defined, and it's easy to check that it is a K-algebra morphism. For injectivity, assume $\Psi(f/g) = 0$ then this means that $\Psi(f/g)$ must vanish on some small open set $V \subset D(g)$ with $V \cap Y \neq \emptyset$. Shrinking it further we can take V to be a distinguished open subset $D(h)$ of some point $a \in V \cap Y$. Then $D(h) \subseteq D(g)$ implies $V(g) \subseteq V(h)$ and so by the Nullstellensatz we find $m \in \mathbb{N}$ with $h^m = g \cdot r \in A(X)$ for some $r \in A(X)$. Since $a \in D(h) \cap Y$ we have $h \notin I(Y)$ and thus also $h^m \notin I(Y)$ and $r \notin I(Y)$ since $I(Y)$ is prime. But then we note

$$0 = \Psi(\frac{f}{g})|_{D(h)} = \frac{f \cdot r}{g \cdot r} = \frac{f \cdot r}{h^m} \in \mathcal{O}_X(D(h)) = A(X)_h \,.$$

By definition, this means that there exists $M \in \mathbb{N}$ with $f \cdot r \cdot h^M = 0 \in A(X)$. But $r \cdot h^M \notin I(Y)$ and applying the definition again, this shows $f/g = 0 \in A(X)_{I(Y)}$, concluding the injectivity part of Ψ.

Conversely, let $[(U, \varphi)] \in \mathcal{O}_{X,Y}$ then again shrinking U further we can assume $U = D(h)$ for $h \notin I(Y)$. But then $\varphi = f/h^m \in \mathcal{O}_X(D(h)) = A(X)_h$ and thus $\varphi = \Psi(f/h^m)$ lies in the image of Ψ, proving its surjectivity.

This concludes the proof that Ψ is an isomorphism of K-algebras as desired.