Exercise Sheet 4

As in class let $\mathbb{P}^1$ be the prevariety obtained by gluing two copies of the affine line $\mathbb{A}^1$ along the isomorphism $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$, $x \to \frac{1}{x}$. By the inclusion of one of the copies, we consider $\mathbb{A}^1$ as an open subprevariety of $\mathbb{P}^1$.

Exercise 1. Which of the following ringed spaces are isomorphic over $\mathbb{C}$?

a) $\mathbb{A}^1$

b) $V(x_1^2 + x_2^2) \subseteq \mathbb{A}^2$

c) $V(x_2 - x_1^2, x_3 - x_1^3) \setminus \{0\} \subseteq \mathbb{A}^3$

d) $V(x_2^2 - x_1^2 x_3 - x_1^3) \setminus \{0\} \subseteq \mathbb{A}_3$

e) $V(x_1 x_2) \subseteq \mathbb{A}^2$

f) $\mathbb{A}^1 \setminus \{1\}$

Solution. The isomorphism classes are a, b=e, c=f, d.

- (c=f) We have that $J = \langle x_2 - x_1^2, x_3 - x_1^3 \rangle$ is a prime ideal with quotient $K[x_1, x_2, x_3]/J \cong K[x_1] = A(\mathbb{A}^1)$. This proves $J = I(V(J))$ and $A(V(J)) = A(\mathbb{A}^1)$ shows $V(J) \cong \mathbb{A}^1$. Under this isomorphism, the point $0 \in V(J)$ maps to $0 \in \mathbb{A}^1$. Removing it gives $V(J) \setminus \{0\} \cong \mathbb{A}^1 \setminus \{0\}$.

- (b=e) The map

  $$F : \mathbb{A}^2 \to \mathbb{A}^2, (x_1, x_2) \mapsto (x_1 + ix_2, x_1 - ix_2)$$

  is a morphism since its components are given by regular functions, and it is in fact an isomorphism (with inverse $(y_1, y_2) \mapsto (1/2(y_1 + y_2), 1/(2i)(y_1 - y_2))$). Moreover we have $F(V(x_1^2 + x_2^2)) = V(x_1 x_2)$, so the varieties from b) and e) are isomorphic.

- Non-isomorphisms: Both $\mathbb{A}^1$ and its open subset $\mathbb{A}^1 \setminus \{0\}$ are irreducible of dimension 1. On the other hand $V(x_1 x_2) = V(x_1) \cup V(x_2)$ is reducible of dimension 1, whereas by Krull's principal ideal theorem, all components of $Y = V(x_2^2 - x_1^2 x_3 - x_1^3)$ are of dimension 2 (which remains true after removing the closed point 0 by Sheet 2, Exercise 5, applied to the components of Y). This shows that among the isomorphism classes a,b,c,d only a and c could possibly coincide.

  It remains to show that $\mathbb{A}^1$ and $\mathbb{A}^1 \setminus \{0\}$ are not isomorphic. If they were, their coordinate rings $R = K[x]$ and $S = K[y]_y = K[y, y^{-1}]$ would have to be isomorphic. Any such isomorphism $K[y, y^{-1}] \to K[x]$ would have to send the invertible element y to an invertible element in $K[x]^\times$. But $K[x]^\times = K^\times$, which shows that the image of y would be a non-zero constant $b \in K^\times$. But then $y - b$ gets sent to zero, giving a contradiction to the assumed isomorphism.

Exercise 2. Let $f : X \to Y$ be a morphism of affine varieties and $f^* : A(Y) \to A(X)$ the corresponding homomorphism of the coordinate rings. Are the following statements true or false?

a) f is surjective if and only if f^* is injective.

b) f is injective if and only if f^* is surjective.

c) If $f : \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism then f is affine linear i.e. of the form $f(x) = ax+b$ for some $a, b \in K$.

d) If $f : \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism then f is affine linear i.e. it is of the form $f(x) = Ax + b$ for some $A \in \mathrm{Mat}(2 \times 2, K)$ and $b \in K^2$.

Solution.

a) It's true that f surjective implies f^* injective. However, the converse is false: consider the inclusion

$$f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \text{ with } f^* : K[x] \to K[y, y^{-1}], x \mapsto y \,. \tag{1}$$

Then f^* is injective, but f is not surjective.

b) It's true that f^* surjective implies f injective: assume $a, b \in X$ with $a \neq b$ satisfy $f(a) = f(b)$. Then choose a function $\varphi \in A(X)$ with $\varphi(a) \neq \varphi(b)$; this is possible since for $X \subseteq \mathbb{A}^n$ there must be a coordinate function $\varphi \in \{x_1, \ldots, x_n\}$ for which $\varphi(a) \neq \varphi(b)$. For $\psi \in A(Y)$ with $f^*\psi = \varphi$ we then have

$$\psi(f(a)) = (f^*\psi)(a) = \varphi(a) \neq \varphi(b) = (f^*\psi)(b) = \psi(f(b))$$

which proves $f(a) \neq f(b)$. However, again the map (1) provides a counter-example for the converse direction: f is injective, but f^* is not surjective.

c) True: by a result from the lecture, any morphism $\mathbb{A}^1 \to \mathbb{A}^1$ is given by a regular function in $\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1) = K[x]$, i.e. a polynomial function. Assume that $f \in K[x]$ is an isomorphism with inverse $g \in K[x]$. Then $f(g(x)) = x$. But if f, g are of degrees d, e, then $f(g(x))$ is of degree $d \cdot e$, as can be checked by writing f, g as a sum of monomial terms and substituting. So d, e are non-negative integers with $\deg(f(g(x)) = d \cdot e = 1 = \deg(x)$, which forces $d = e = 1$. Thus $f(x) = ax + b$ is linear as claimed.

d) False: given any $h \in K[x]$, the function

$$f : \mathbb{A}^2 \to \mathbb{A}^2, f(x_1, x_2) = (x_1, x_2 + h(x_1))$$

is an isomorphism with inverse

$$g : \mathbb{A}^2 \to \mathbb{A}^2, g(x_1, x_2) = (x_1, x_2 - h(x_1)) \,.$$

However, in general it is not an affine linear function f as in the question.

Exercise 3. Prove the following statements:

a) Every morphism $\mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^1 \to \mathbb{P}^1$.

b) Not every morphism $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^2 \to \mathbb{P}^1$.

c) Every morphism $\mathbb{P}^1 \to \mathbb{A}^1$ is constant.

Solution. Let $X_1 = X_2 = \mathbb{A}^1$ be the two copies of the affine line covering $\mathbb{P}^1 = X_1 \cup X_2$.

a) Let $f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$ be a morphism. If f is constant equal to the map $z \mapsto \infty$, then it can be extended to a constant morphism. Otherwise, let $U_1 = f^{-1}(X_1) \subseteq \mathbb{A}^1 \setminus \{0\}$, then since f is not constant equal to ∞ we have that U_1 is a non-empty open subset of $\mathbb{A}^1 \setminus \{0\}$. By our knowledge of the open subsets of $\mathbb{A}^1$ we then have $U_1 = \mathbb{A}^1 \setminus \{0, a_1, \ldots, a_r\}$. Then $f|_{U_1} : U_1 \to X_1 = \mathbb{A}^1$ is a morphism, so it is given by a regular function on U_1. But $U_1 = D(g)$ is a distinguished affine open subset, for $g(x) = x(x - a_1) \cdots (x - a_r)$ and so $f(x) = h(x)/g(x)^m$ for some $m \in \mathbb{N}$. We distinguish two cases:

- If the order of h at 0 is less or equal than m, the regular function f does not actually have a pole at 0 (after clearing the term x^m in the denominator) and thus extends to a regular function $\widetilde{f}$ on $U_1 \cup \{0\} = \mathbb{A}^1 \setminus \{a_1, \ldots, a_r\}$. The functions $f, \widetilde{f}$ agree on the overlaps of their domains of definition, and both define morphisms to $X_1 \subseteq \mathbb{P}^1$. From the lecture, we know that they can then be glued to a morphism on $\mathbb{A}^1$ extending the original morphism f.
- If the order of h at 0 is greater than m, we have that $\widetilde{f} = 1/f$ is a regular function on some open subset $\widetilde{U} \subseteq U_1$ which can be extended over 0. Checking the gluing definition of $\mathbb{P}^1$, one verifies that the functions

$$f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1 \text{ and } \widetilde{f} : \widetilde{U} \cup \{0\} \to X_2 \subseteq \mathbb{P}^1$$

  agree on their overlap and thus glue to a function $\mathbb{A}^1 \to \mathbb{P}^1$ as desired.

b) Consider the open sets $U_1 = D(x_1)$, $U_2 = D(x_2)$ covering $\mathbb{A}^2 \setminus \{0\}$. Then the functions

$$U_1 \to X_1 \subseteq \mathbb{P}^1, (x_1, x_2) \mapsto x_2/x_1 \text{ and } U_2 \to X_2 \subseteq \mathbb{P}^1, (x_1, x_2) \mapsto x_1/x_2$$

agree on their overlaps and glue to a function $f : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$. Assume that f could be extended to a morphism $f : \mathbb{A}^2 \to \mathbb{P}^1$. Consider the points $0, \infty \in \mathbb{P}^1$, then $f^{-1}(0), f^{-1}(\infty)$ would be disjoint, Zariski closed subsets of $\mathbb{A}^2$. But by definition, we have that f sends $V(x_2) \setminus \{0\}$ to 0 and $V(x_1) \setminus \{0\}$ to ∞. Thus the same must be true for their Zariski closures $V(x_2), V(x_1)$, but then we get a contradiction, since $0 \in V(x_1) \cap V(x_2)$ must be sent both to 0 and to ∞.

c) The restriction of the morphism $f : \mathbb{P}^1 \to \mathbb{A}^1$ to X_1, X_2 is given by $f_1 \in \mathcal{O}_{X_1}(X_1) = K[x]$ and $f_2 \in \mathcal{O}_{X_2}(X_2) = K[w]$. By the compatibility on the overlap, these two polynomials must satisfy $f_1(1/x) = f_2(x) \in \mathcal{O}_{X_1}(X_1 \setminus \{0\}) = K[x]_x$. Note that $f_1(1/x)$ is a linear combination of $1, 1/x, 1/x^2, \ldots$ and $f_2(x)$ is a linear combination of $1, x, x^2, \ldots$. However, the rational functions $\ldots, 1/x^2, 1/x, 1, x, x^2, \ldots$ are all linearly independent in $K[x]_x$, so this is only possible if f_1, f_2 are constant (and equal). This shows that f is constant itself, finishing the proof.

Exercise 4. If X and Y are affine varieties we have seen that there is a bijection

$$\{\text{morphisms } X \to Y\} \overset{1:1}{\longleftrightarrow} \{\text{K-algebra homomorphisms } \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)\}, \quad f \mapsto f^*.$$

a) Does this statement still hold if X is an arbitrary prevariety (but Y is still affine)?

b) Does this statement still hold if Y is an arbitrary prevariety (but X is still affine)?

Solution.

a) Yes: choose a cover $\{U_i : i \in I\}$ of X by affine varieties U_i, and cover each intersection $U_i \cap U_j$ by further affine varieties $\{V_{i,j,k} : k \in J_{i,j}\}$. Then, any morphism $X \to Y$ gives a collection of morphisms $f_i : U_i \to Y$ by restriction. Conversely, such a collection of morphisms glues together if they have compatible restrictions $f_i|_{V_{i,j,k}} = f_j|_{V_{i,j,k}}$. This follows from the gluing property of morphisms together with the fact that the $V_{i,j,k}$ cover $U_i \cap U_j$. To summarize, we have a bijection:

$$\{\text{morphisms } X \to Y\} \overset{1:1}{\longleftrightarrow} \{\text{morphisms } f_i : U_i \to Y : f_i|_{V_{i,j,k}} = f_j|_{V_{i,j,k}}\} \qquad (2)$$

But on the right hand side of (2) we note that all morphisms $U_i \to Y$ and $V_{i,j,k} \to Y$ that appear now have affine varieties as domain and target. So we can convert the known correspondence and convert them to K-algebra morphisms $g_i = f_i^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(U_i)$ such that the compositions of g_i, g_j with the restriction maps to $\mathcal{O}_X(V_{i,j,k})$ agree for all $k \in J_{i,j}$. Given a section $\varphi \in \mathcal{O}_Y(Y)$, the sections $g_i(\varphi) \in \mathcal{O}_X(U_i)$ and $g_j(\varphi) \in \mathcal{O}_X(U_j)$ agree on the overlap $U_i \cap U_j$ (here we use the uniqueness part of the sheaf property of $\mathcal{O}_X$ for the cover $U_i \cap U_j = \bigcup_k V_{i,j,k}$). Thus there exists a unique section $\psi \in \mathcal{O}_X(X)$ with $\psi|_{U_i} = g_i(\varphi)$, where we use the existence and uniqueness part of the sheaf property for the cover $X = \bigcup_i U_i$. We *define* a map $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ by sending φ to ψ as constructed above. Then one checks that this is a K-algebra homomorphism. All operations above (passing from $f : X \to Y$ to the collection $g_i = f_i^*$ and then gluing together again to g) were bijections, which finishes the proof.

b) No: take $X = \mathbb{A}^1$ and $Y = \mathbb{P}^1$, then we have seen above that $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = K$, so the only K-algebra homomorphism $K = O_Y(Y) \to O_X(X) = K[x]$ is given by the canonical inclusion $K \to K[x]$. However, we know that there are at least two different morphisms $\mathbb{A}^1 \to \mathbb{P}^1$ given by the inclusions of X_1, X_2. Thus no bijection as above is possible.