Exercise Sheet 5

Exercise 1.

a) Compute explicit generators for the ideal $I(a) \subseteq K[x_0, \dots, x_n]$ of an arbitrary point $a \in \mathbb{P}^n$.

b) Let $X = V(x_1^2 - x_2^2 - 1, x_3 - x_1) \subset \mathbb{A}^3_{\mathbb{C}}$. What are the points at infinity of the projective closure $\overline{X} \subset \mathbb{P}^3_{\mathbb{C}}$, i.e., the points in $\overline{X} \backslash X$?

Solution.

a) We claim that the ideal $I(a)$ has generators as follows:

$$I(a) = \langle x_i a_j - x_j a_i : i, j = 0, \dots, n \rangle =: J \subseteq K[x_0, \dots, x_n]\,. \tag{1}$$

The inclusion $I(a) \supseteq J$ is clear from plugging $x = a$ into the generators of J. In fact, these generators are precisely the 2×2-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & \dots & x_n \\ a_0 & a_1 & \dots & a_n \end{pmatrix} ,$$

and so they vanish if and only if the matrix M has rank (at most) 1. This is equivalent to the rows x, a of the matrix to be linearly dependent, or $[x] = [a] \in \mathbb{P}^n$. This proves $V_p(J) = \{a\}$. By the projective Nullstellensatz, it then suffices to show that J is a radical ideal, to conclude $I(a) = \sqrt{J} = J$.

To prove this, we compute $R = K[x_0, \dots, x_n]/J$ and show that it has no non-trivial nilpotent elements. Assume without loss of generality that $a_0 \neq 0$, then we have $[x_j] = [x_0 a_j / a_0] \in R$ from one of the generators of J. Using this to eliminate the variables $x_1, \dots, x_n$ we get an isomorphism

$$K[x_0] \xrightarrow{\sim} R, x_0 \mapsto [x_0]\,.$$

The inverse of this map is given by

$$R \to K[x_0], x_j \mapsto x_0 \frac{a_j}{a_0}\,.$$

One double-checks that all generators of J map to zero under this map (so it's well-defined) and it sends $x_0 \mapsto x_0$, so composing it with the map $K[x_0] \to R$ gives the identity on $K[x_0]$.

By this argument, we conclude that $R \cong K[x_0]$ is reduced as claimed.

b) First note that $X \subseteq V(x_3 - x_1) \cong \mathbb{A}^2_{\mathbb{C}} \subseteq \mathbb{A}^3_{\mathbb{C}}$ is contained in a linear subspace of codimension 1. The projective closure of this copy of $V_a(x_3 - x_1)$ is $V_p(x_3 - x_1) \subseteq \mathbb{P}^3_{\mathbb{C}}$ which is the image of the closed embedding

$$\varphi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}, (x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : x_2 : x_1)$$

obtained by setting $x_3 = x_1$ in projective coordinates. By basic topology, the closure of X in $\mathbb{P}^3_{\mathbb{C}}$ agrees with its closure in $V_p(x_3 - x_1) \cong \mathbb{P}^2_{\mathbb{C}}$. Thus

$$\overline{X} = \varphi(\overline{Y}) \text{ for } Y = V_a(x_1^2 - x_2^2 - 1) \subseteq \mathbb{A}^2_{\mathbb{C}}\,.$$

But the ideal of Y is principal, and in that case we saw that the projective closure $\overline{Y}$ is cut out by the homogenization of the equation, so $\overline{Y} = V_p(x_1^2 - x_2^2 - x_0^2)$. Then $\overline{Y} \setminus Y$ is obtained by imposing the additional equation $x_0 = 0$, leading to $(x_1 - x_2) \cdot (x_1 + x_2) = 0$ and the solution points $\overline{Y} \setminus Y = \{(0 : 1 : 1), (0 : 1 : -1)\}$. Applying the map φ we obtain $\overline{X} \setminus X = \{(0 : 1 : 1 : 1), (0 : 1 : -1 : 1)\}$.

Exercise 2. A line in $\mathbb{P}^n$ is a linear subspace of dimension 1.

a) Let $L_1, L_2 \subset \mathbb{P}^3$ be two disjoint lines and let $a \in \mathbb{P}^3 \setminus (L_1 \cup L_2)$. Show that there is a unique line $L \subset \mathbb{P}^3$ through a that intersects both L_1 and L_2.
Hint: Think about the corresponding cones in $\mathbb{A}^4$.

b) Is the corresponding statement for lines and points in $\mathbb{A}^3$ true as well?

Solution.

a) The cones $C(L_1), C(L_2) \subseteq K^4$ are 2-dimensional linear subspaces. The assumption that L_1, L_2 are disjoint translates to these subspaces intersecting only at $\{0\}$, so in fact $K^4 = C(L_1) \oplus C(L_2)$ as a vector space. By choosing bases v_1, w_1 and v_2, w_2 of $C(L_1), C(L_2)$, we obtain a basis v_1, w_1, v_2, w_2 of K^4, and applying a (projective) linear automorphism, we can assume that $C(L_1) = \langle e_1, e_2 \rangle$ and $C(L_2) = \langle e_3, e_4 \rangle$. Indeed, such a projective automorphism sends lines to lines, so the question can be solved after moving L_1, L_2 to the positions above.

Assume that the intersection points of a line L through $a = (a_0 : a_1 : a_2 : a_3)$ with L_1, L_2 are given by $(c_0 : c_1 : 0 : 0)$ and $(0 : 0 : c_2 : c_3)$. Thus the line L is given by

$$L = \{(\lambda c_0 : \lambda c_1 : \mu c_2 : \mu c_3) : (\lambda : \mu) \in \mathbb{P}^1\}\,.$$

The condition $a \in L$ is then equivalent to $(c_0 : c_1) = (a_0 : a_1) \in \mathbb{P}^1$ and $(c_2 : c_3) = (a_2 : a_3) \in \mathbb{P}^1$, hence this is the unique line satisfying the conditions above. Note here that the points $(a_0 : a_1), (a_2, a_3) \in \mathbb{P}^1$ are well-defined since $a \notin L_1 \cup L_2 = V(x_2, x_3) \cup V(x_0, x_1)$.

b) No: take $L_1 = \{(t, 0, 0) : t \in K\}$ and $L_2 = \{(0, t, 1) : t \in K\}$ as well as $a = (1, 0, 1)$. If there was a line L_0 through a meeting L_1, L_2, then the projective closure $L = \overline{L}_0$ would meet the projective closures $\overline{L}_1, \overline{L_2}$ and still go through a. But the unique line in $\mathbb{P}^3$ through a meeting the projective closures of L_1, L_2 is the projective closure $L = \{(s : t : 0 : s) : (s : t) \in \mathbb{P}^1\}$ of $L_0 = \{(t, 0, 1) : t \in K\}$. The line L meets L_2 in $(0, 0, 1)$, for $(s : t) = (1 : 0)$, goes through a at $(s : t) = (1 : 1)$ and meets L_1 at $(0 : 1 : 0 : 0)$ for $(s : t) = (0 : 1)$. Since this last intersection point happens outside of $\mathbb{A}^3 \subseteq \mathbb{P}^3$, the statement of part a) is false in this case when applied to $\mathbb{A}^3$ instead of $\mathbb{P}^3$.

Exercise 3.

a) Prove that a graded ring R is an integral domain if and only if for all homogeneous elements $f, g \in R$ with $fg = 0$ we have $f = 0$ or $g = 0$.

b) Show that a projective variety X is irreducible if and only if its homogeneous coordinate ring $S(X)$ is an integral domain.

Solution.

a) The given condition without the assumption of f, g being homogeneous is the definition of an integral domain, so one direction is obvious. Assume conversely that we have $f = f_0 + \ldots + f_d$ and $g = g_0 + \ldots + g_e \in R$ with $f \cdot g = 0$. We prove that $f = 0$ or $g = 0$ by induction on $d+e$. The base case $d+e = 0$ follows by assumption since then f, g are homogeneous. In the induction step, note that the degree $d+e$ part of $f \cdot g$ is precisely given by $f_d \cdot g_e$, hence again using the assumption we have $f_d = 0$ or $g_e = 0$. This reduces us to the case where the sum of the degrees of f and g are at most $d + e - 1$, completing the induction.

b) If $S(X)$ is not an integral domain, by part a) we find $f, g \in S(X)$ homogeneous with $f, g \neq 0$ but $f \cdot g = 0$. But then for $X_1 = V_{X,p}(f)$ and $X_2 = V_{X,p}(g)$ we have $X_1 \cup X_2 = V_{X,p}(f \cdot g) = X$ but $X_i \subsetneq X$ since $f, g \notin I_p(X)$.

Conversely, assume that $S(X)$ is an integral domain and that we had a cover $X = X_1 \cup X_2$ by two closed subsets neither of which is all of X. Let $a_1, a_2 \in X$ with $a_1 \in X_1 \setminus X_2$ and $a_2 \in X_2 \setminus X_1$. Then there must be a homogeneous $f_1 \in I_X(X_1)$ with $f_1(a_2) \neq 0$, since $a_2 \notin X_1 = V_{X,p}(I_X(X_1))$. Similarly, we find $f_2 \in I_X(X_2)$ with $f_2(a_1) \neq 0$. Then clearly $f_1, f_2 \neq 0 \in S(X)$, since there are points of X where they don't vanish, but $f_1 \cdot f_2$ vanishes on $X_1 \cup X_2 = X$, and thus $f_1 \cdot f_2 = 0 \in S(X)$. This gives a contradiction to $S(X)$ being an integral domain.

Exercise 4. In this exercise we want to show that an intersection of projective varieties is never empty unless one would expect it to be empty for dimensional reasons — so, e.g., the phenomenon of parallel non-intersecting lines in the plane does not occur in projective space.

a) Let $X, Y \subset \mathbb{A}^n$ be pure-dimensional affine varieties. Show that every irreducible component of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$.
Hint: Use diagonals.

b) Now let $X \subseteq \mathbb{P}^n$ be a non-empty projective variety. Prove that the dimension of the cone $C(X) \subset \mathbb{A}^{n+1}$ is $\dim X + 1$.

c) Let $X, Y \subset \mathbb{P}^n$ be projective varieties with $\dim X + \dim Y \geq n$. Show that $X \cap Y \neq \emptyset$.

Solution.

a) For the diagonal $\Delta_{\mathbb{A}^n} \subseteq \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ we have $X \cap Y \cong X \times Y \cap \Delta_{\mathbb{A}^n}$. To see this one can e.g. use that the map $\delta : \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n, x \mapsto (x, x)$ is a closed embedding with image Δ and $X \cap Y = \delta^{-1}(X \times Y)$.

However, we know that $\dim X \times Y = \dim X + \dim Y$ and that

$$\Delta = V(x_1 - y_1, \ldots, x_n - y_n) \subseteq \mathbb{A}^{2n}\,.$$

So we obtain $X \cap Y$ by starting with the space $X \times Y$ of dimension $\dim X + \dim Y$ and imposing the n equations $x_i - y_i$ for $i = 1, \ldots, n$. By Krull's principal ideal theorem applied inductively, the dimension of each irreducible component after imposing the first n' conditions is at least $\dim X + \dim Y - n'$, so for $n' = n$ we obtain the desired result.
Note: It is possible that the dimension does not go down for all components when imposing the n'th condition, in the case where this condition vanishes identically on some components of the previous intersection. This is the reason why we only have an inequality in the statement.

b) Since $X \neq \emptyset$, we have $C(X) \neq \{0\}$ is of dimension at least 1, and hence $\dim C(X) = \dim C(X) \setminus \{0\}$.
Proof: Assume that $C(X) = C_1 \cup \ldots \cup C_s$ is an irreducible decomposition, then $\dim C(X) = \max_i \dim C_i$. Since $C(X) \neq \{0\}$, it contains at least one non-zero element and thus (since it is a cone) an entire line, proving that $\dim C(X) \geq 1$. Now consider $C(X) \setminus \{0\}$, then its irreducible decomposition is given by

$$C(X) \setminus \{0\} = (C_1 \setminus \{0\}) \cup \ldots \cup (C_s \setminus \{0\})\,.$$

Each $C_i \setminus \{0\} \subseteq C_i$ is a non-empty open subset (if it was empty, then $C_i = \{0\}$, but this is a contradiction since 0 is contained in some of the other C_i by the cone property). So by Sheet 2, Exercise 5 b) it has the same dimension as C_i. This shows

$$\dim C(X) \setminus \{0\} = \max_i \dim(C_i \setminus \{0\}) = \max_i \dim C_i = \dim C(X)\,.$$

For $U_i \subseteq \mathbb{P}^n$ and $V_i \subseteq \mathbb{A}^{n+1}$ the loci where the i-th coordinate does not vanish, we have that X is covered by the open subsets $X \cap U_i$ and $C(X) \setminus \{0\}$ is covered by $C(X) \cap V_i$. Since the dimension is the supremum of dimensions of an open cover, it suffices to show that

$$\dim(X \cap U_i) + 1 = \dim(C(X) \cap V_i)\,.$$

Identifying U_i with the locus $V(x_i - 1) \subseteq \mathbb{A}^{n+1}$ the above equality just follows from the isomorphism

$$(X \cap U_i) \times (\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\sim} C(X) \cap V_i$$

given by

$$((x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n), t) \mapsto (tx_0, \ldots, tx_{i-1}, t, tx_{i+1}, \ldots, x_n)\,.$$

Its inverse is given by

$$(y_0, \ldots, y_n) \mapsto ((y_0/y_i, \ldots, y_n/y_i), y_i)\,.$$

We conclude since $\dim(X \cap U_i) \times (\mathbb{A}^1 \setminus \{0\}) = \dim(X \cap U_i) + \dim(\mathbb{A}^1 \setminus \{0\}) = \dim(X \cap U_i) + 1$.

c) By decomposing X, Y into irreducible components and choosing ones which have maximal dimension, it suffices to show the statement when X, Y are irreducible themselves. Then an adaption of the argument in b) shows that $C(X), C(Y)$ are irreducible as well. Moreover, we have $C(X \cap Y) = C(X) \cap C(Y)$ and the origin 0 is contained in that intersection. Let Z be an irreducible component of $C(X) \cap C(Y)$ containing the origin. Then by part a) we know it has dimension at least

$$\dim(C(X)) + \dim(C(Y)) - (n+1) = \dim(X) + 1 + \dim(Y) + 1 - n - 1 \geq 1\,,$$

where the first equality uses part b). Thus we have that $Z \neq \{0\}$ and for any point $z \in Z \setminus \{0\}$ we have that $[z] \in X \cap Y \subseteq \mathbb{P}^n$ is a point in the intersection, finishing the proof.