Exercise Sheet 6

Exercise 1. Let $m, n \in \mathbb{N}_{>0}$. Prove:

a) If $f : \mathbb{P}^n \to \mathbb{P}^m$ is a morphism and $X \subset \mathbb{P}^m$ a hypersurface then every irreducible component of $f^{-1}(X)$ has dimension at least $n-1$.
Hint: Use that locally on an affine cover of $\mathbb{P}^m$, the hypersurface X is cut out by a single equation.

b) Show that any morphism $f : \mathbb{P}^n \to \mathbb{A}^m$ is constant.

c) If $n > m$ then every morphism $f : \mathbb{P}^n \to \mathbb{P}^m$ is constant.
Hint: Consider the preimages of the hyperplanes $V(x_i) \subseteq \mathbb{P}^m$ under f.

d) $\mathbb{P}^n \times \mathbb{P}^m$ is not isomorphic to $\mathbb{P}^{n+m}$.

Solution.

a) Let $\mathbb{P}^m = U_0 \cup \ldots \cup U_m$ be the standard cover of the target of f. Then $f^{-1}(X)$ is covered by the open sets $f^{-1}(X \cap U_i)$ for $i = 0, \ldots, m$. Thus it suffices to check that every irreducible component of $f^{-1}(X \cap U_i)$ is of dimension at least $n-1$.

We distinguish two cases: if $f^{-1}(U_i) = \emptyset$, then the statement is trivially satisfied since $f^{-1}(X \cap U_i) = \emptyset$. Otherwise, we know that $f^{-1}(U_i) \subseteq \mathbb{P}^n$ is a non-empty open subset, and hence irreducible of dimension n. Let $\{V_j : j \in J\}$ be an open cover of $f^{-1}(U_i)$ by affine varieties, which are necessarily irreducible of dimension n. Consider the map
$$f|_{V_j} : V_j \to U_i \cong \mathbb{A}^m .$$
We know that $X \cap U_i \subseteq U_i \cong \mathbb{A}^m$ is a hypersurface. Since the coordinate ring of $\mathbb{A}^m$ is a UFD, its ideal is principal, so $X \cap U_i = V(F)$ for $F \in K[x_1, \ldots, x_m]$, where $x_1, \ldots, x_m$ are coordinates on U_i. Then
$$f^{-1}(X) \cap V_j = V_{V_j}((f|_{V_j})^* F)$$
is the vanishing locus of the function $(f|_{V_j})^* F \in A(V_j)$. By Krull's principal ideal theorem, every irreducible component of this vanishing locus has dimension at least $\dim(V_j) - 1 = n - 1$, finishing the proof.

b) We have seen that the morphisms $X \to \mathbb{A}^m$ from any prevariety X to affine space $\mathbb{A}^m$ are given by a collection of m regular functions on X. Since $X = \mathbb{P}^n$ is complete, all regular functions are constant, and so the map f is constant as well.

c) Assume on the contrary that $f : \mathbb{P}^n \to \mathbb{P}^m$ is a non-constant map with $n > m$. First, we claim that each of the hyperplanes $V(x_i) \subseteq \mathbb{P}^m$ satisfies $H_i := f^{-1}V(x_i) \neq \emptyset$. Otherwise, the map f would factor through the complement $\mathbb{P}^m \setminus V(x_i) \cong \mathbb{A}^m$. By part b) such a map is then constant, giving a contradiction.

Thus, by part a) we have that all preimages $H_i \subseteq \mathbb{P}^n$ are non-empty and all of their components have dimension at least $n-1$. For two projective varieties X, Y in $\mathbb{P}^n$ whose dimension sums up to at least n, we have proven that their intersection is non-empty, and the proof also implied that its dimension is at least $\dim X + \dim Y - n$. Applying this result iteratively, we see that the intersection $H_0 \cap \ldots \cap H_k$ is non-empty of dimension at least $n-k$. Specializing to $k = m$, we have that $H_0 \cap \ldots \cap H_m$ is of dimension at least $n - (m+1) \geq 0$ and non-empty. But

$$H_0 \cap \ldots \cap H_m = f^{-1}(V(x_0)) \cap \ldots \cap f^{-1}(V(x_m)) = f^{-1}(V(x_0, \ldots, x_m)) = f^{-1}(\emptyset) = \emptyset\,,$$

which gives a contradiction.

d) The projection map $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ is not constant. Since $n + m > n$ this would give a contradiction if we had an isomorphism $\mathbb{P}^n \times \mathbb{P}^m \cong \mathbb{P}^{n+m}$.

Exercise 2. Let us say that $n+2$ points in $\mathbb{P}^n$ are in general position if for any $n+1$ of them their representatives in K^{n+1} are linearly independent.
Now let $a_1, \ldots, a_{n+2}$ and $b_1, \ldots, b_{n+2}$ be two sets of points in $\mathbb{P}^n$ in general position.

a) Show that the collection $A_1 = e_0 = (1 : 0 : \ldots : 0)$, $A_2 = e_1 = (0 : 1 : 0 : \ldots : 0)$, $\ldots$, $A_{n+1} = e_n = (0 : \ldots : 0 : 1)$, $A_{n+1} = (1 : 1 : \ldots : 1)$ is in general position.

b) Show that there is an isomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ with $f(a_i) = b_i$ for all $i = 1, \ldots, n+2$.

Solution.

a) The vectors $e_0, \ldots, e_n$ are the standard basis of K^{n+1} and thus linearly independent. On the other hand, given $e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n, A_{n+1}$ we obtain this standard basis by subtracting the sum of the first n vectors in the collection from A_{n+1} (since this gives the missing vector e_i). This corresponds to a row-operation in the matrix with rows $e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n, A_{n+1}$ and since this preserves the rank, we again have that the vectors are linearly independent.

b) It suffices to prove the claim for $a_i = A_i$ as in part a). Indeed, if the isomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ sends A_i to a_i and the isomorphism $g : \mathbb{P}^n \to \mathbb{P}^n$ sends A_i to b_i, then $g \circ f^{-1}$ sends a_i to b_i.

Since $b_1, \ldots, b_{n+1}$ are linearly independent, they form a basis of K^{n+1} and so for any numbers $\lambda_1, \ldots, \lambda_{n+1} \in K^\times$, the matrix M with columns given by $\lambda_1 b_1, \ldots, \lambda_{n+1} b_{n+1}$ is invertible, and thus gives an isomorphism $\mathbb{P}^n \to \mathbb{P}^n, x \mapsto Mx$. By definition, this sends $A_i = e_{i-1}$ to $[\lambda b_i] = [b_i] \in \mathbb{P}^n$. On the other hand, it sends A_{n+1} to the vector

$$[\lambda_1 b_1 + \ldots + \lambda_{n+1} b_{n+1}] \in \mathbb{P}^n\,.$$

On the other hand, there is a unique linear combination of the basis elements $b_1, \ldots, b_{n+1}$ giving b_{n+2}. In this linear combination, none of the coefficients λ_i of

b_i can vanish, since then the $n+1$ vectors $\{b_1, \ldots, b_{n+2}\} \setminus \{b_i\}$ would be linearly dependent, giving a contradiction to the assumption of being in general position. Taking the λ_i above to be the coefficients of this unique linear combination, we indeed have $M \cdot A_{n+2} = [b_{n+2}]$.

Exercise 3. Show by example that the homogeneous coordinate ring of a projective variety is not invariant under isomorphisms i.e., that there are isomorphic projective varieties X, Y such that the graded K-algebras $S(X)$ and $S(Y)$ are not isomorphic.

Solution. We have seen in class (when discussing the projection from a point) that the map

$$f : X = \mathbb{P}^1 \to V_p(x_0x_2 - x_1^2) =: Y \subseteq \mathbb{P}^2, (s : t) \mapsto (s^2 : st : t^2)$$

is an isomorphism. The degree 1 part $S(X)_1 = \langle s, t\rangle$ is of dimension 2. On the other hand, the degree 1 part $S(Y)_1 = \langle x_0, x_1, x_2\rangle$ has three generators, and they are linearly independent, since all nonzero elements of $I(Y) = \langle x_0x_2 - x_1^2\rangle$ have degree at least 2 and $S(Y) = K[x_0, x_1, x_2]/I(Y)$. Thus $\dim S(Y)_1 = 3 > \dim S(X)_1 = 2$, which shows $S(X) \ncong S(Y)$ as graded rings (and also not as K-algebras, since $S(Y)$ cannot be generated by two elements over K).

Exercise 4. A conic over a field of characteristic not equal to 2 is an irreducible curve in $\mathbb{P}^2$ of degree 2.

a) Using the coefficients of quadratic polynomials show that the set of all conics can be identified with an open subset U of $\mathbb{P}^5$. (One says that U is a moduli space for conics.)

b) Given a point $p \in \mathbb{P}^2$ show that the subset of U consisting of all conics passing through p is the zero locus of a linear equation in the homogeneous coordinates of $U \subset \mathbb{P}^5$.

c) Given 5 points in $\mathbb{P}^2$, no three of which lie on a line, show that there is a unique conic passing through all these points.

Solution.

a) A quadratic polynomial on $\mathbb{P}^2$ is of the form

$$f = a_{00}x_0^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 \in K[x_0, x_1, x_2]\,.$$

Since its vanishing set $C = V(f) \subseteq \mathbb{P}^2$ does not change when scaling f, this vanishing set is uniquely determined by the vector

$$a = (a_{00} : a_{01} : a_{02} : a_{11} : a_{12} : a_{22}) \in \mathbb{P}^5$$

describing the coefficients of $f = f_a$. Moreover, the vanishing set C is an irreducible curve of degree 2 if and only if f is irreducible (and in this case we have $I(C) = \langle f\rangle$, so f up to scaling is uniquely determined by C).

To conclude, we want to show that inside $\mathbb{P}^5$, the locus U of points a such that f_a is irreducible is an open set. If f had a factorization $f = f_1 \cdot f_2$, then necessarily f_1, f_2 would have to be homogeneous polynomials, and in order for the factorization to not be trivial, they would both have to be linear. Thus we see that the complement $\mathbb{P}^5 \setminus U$ is the image of the morphism

$$\Phi : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$$
$$(\underbrace{b_0x_0 + b_1x_1 + b_2x_2}_{f_1}, \underbrace{c_0x_0 + c_1x_1 + c_2x_2}_{f_2}) \mapsto f = f_1 \cdot f_2 \,.$$

Here again we identify $\mathbb{P}^2$ with the moduli space of linear polynomials (via the coefficient vectors $(b_0 : b_1 : b_2)$ and $(c_0 : c_1 : c_2)$ of f_1, f_2). One checks that Φ is a morphism, e.g. on an open cover of $\mathbb{P}^5$. Then since $\mathbb{P}^2$ is projective, also $\mathbb{P}^2 \times \mathbb{P}^2$ is projective, and thus complete. Hence the image of Φ is closed, and so its complement U is open.
Note: Alternatively, one can show that the set of reducible quadratic polynomials is closed by using that they are described by the vanishing of a suitable discriminant function.

b) For $p = (y_0 : y_1 : y_2) \in \mathbb{P}^2$ the condition that $p \in C = V(f_a)$ is just given by

$$a_{00}y_0^2 + a_{01}y_0y_1 + a_{02}y_0y_2 + a_{11}y_1^2 + a_{12}y_1y_2 + a_{22}y_2^2 \,.$$

Since the y_i are fixed numbers, this is just a linear condition on the coordinates a_{ij} on $\mathbb{P}^5$.

c) Intuitively, for each of the points $p_1, \ldots, p_5$, the condition that $p_i \in V(f_a)$ gives a linear subspace of U of codimension 1. Intersecting these five subspaces, we expect to obtain a linear space of codimension 5, i.e. a single point.

To make this precise: using Exercise 2, we see that the condition of no three points lying on a line exactly means that no three representatives of the points are linearly dependent. Thus the points p_i are in general position and we can find an isomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ sending p_1, p_2, p_3, p_4 to $A_1 = (1 : 0 : 0)$, $A_2 = (0 : 1 : 0)$, $A_3 = (0 : 0 : 1)$ and $A_4 = (1 : 1 : 1)$. Any such isomorphism sends a conic C through the p_i to a conic $f(C)$ through A_1, A_2, A_3, A_4 and $p = f(p_5) = (y_0 : y_1 : y_2)$, and this correspondence is a bijection.

So it suffices to solve the problem for the points A_1, A_2, A_3, A_4, p. Looking at the equation above, the conditions $A_1, A_2, A_3 \in V(f_a)$ mean that $a_{00} = a_{11} = a_{22} = 0$. Moreover, the condition $A_4 \in V(f_a)$ implies that $a_{01} + a_{02} + a_{12} = 0$. Solving this last equation for $a_{12} = -a_{01} - a_{02}$ the set of quadratic equations whose vanishing set locus contains $A_1, \ldots, A_4$ is given by f of the form

$$f = a_{01}(x_0x_1 - x_1x_2) + a_{02}(x_0x_2 - x_1x_2)$$

We claim that the additional condition $f(p) = f(y_0, y_1, y_2) = 0$ gives one more linear condition on the coefficients a_{01}, a_{02} and thus we obtain a unique solution for f up to scaling, so that there is a unique quadratic equation f satisfied by the five points. If $f(p) = 0$ was *not* an additional condition, then $f(p)$ would have to vanish as a polynomial in a_{01}, a_{02} and thus we would have

$$0 = y_0y_1 - y_1y_2 = (y_0 - y_2)y_1 \text{ and } 0 = y_0y_2 - y_1y_2 = (y_0 - y_1)y_2 \,.$$

Going through the cases, these two equations can only be satisfied if $p = (y_0 : y_1 : y_2) \in \{A_1, \ldots, A_4\}$, which gives a contradiction.

We have thus proved that there is a unique quadratic equation f (up to scaling) satisfied by the five points A_1, A_2, A_3, A_4, p. If f was not irreducible, it would decompose as the product of two linear polynomials $f = f_1 f_2$. But then the five points are contained in the union $L_1 \cup L_2 = V(f_1) \cup V(f_2)$ of lines, so one of the two lines must contain at least three points. This gives a contradiction to the assumption.

Fancier argument: Returning to our original proof idea, let $L_1, \ldots, L_5 \subseteq \mathbb{P}^5$ be the five linear hyperplanes describing quadratic equations going through the points $p_1, \ldots, p_5$. Then for $D = L_1 \cap \ldots L_5$ we know that D is a linear subspace of $\mathbb{P}^5$ and the conics we look for are exactly the points in $D \cap U$. We claim that D must consist of a single point, and this point must lie in U.

First, the same argument as above shows that any quadratic equation f satisfied by the five points cannot decompose into two linear factors, by the assumption that no three points p_i lie on a line. This shows that D does not intersect the space of reducible quadratic equations, and thus is contained in U. Using a bit more work than above, one can show however that the complement $Y = \mathbb{P}^5 \setminus U$ is actually a projective hypersurface, i.e. of codimension 1. If D had dimension at least 1, there would have to be an intersection point in $D \cap Y$ since D, Y must intersect in $\mathbb{P}^5$ by dimension reasons (as seen in a previous exercise). This shows that D is of dimension 0, and thus a single point (since it is also a linear space). The inclusion $D \subseteq U$ that we already proved above then finishes the argument.