Exercise Sheet 7

Exercise 1. Let $X \subseteq \mathbb{P}^3$ be the degree-3 Veronese embedding of $\mathbb{P}^1$, i.e., the image of the morphism

$$\mathbb{P}^1 \to \mathbb{P}^3, \quad (x_0 : x_1) \to (y_0 : y_1 : y_2 : y_3) = (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3).$$

Moreover, let $a = (0 : 0 : 1 : 0) \in \mathbb{P}^3$ and $L = V(y_2) \subseteq \mathbb{P}^3$ and let f be the projection from a to L.

a) Determine an equation of the curve $f(X)$ in $L \cong \mathbb{P}^2$.

b) Is $f : X \to f(X)$ an isomorphism onto its image?

Solution.

a) First, we claim that the projection $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is given by $f(y_0 : y_1 : y_2 : y_3) = (y_0 : y_1 : y_3)$. Indeed, the unique line through a and $(y_0 : y_1 : y_2 : y_3)$ is given by

$$\{(sy_0 : sy_1 : sy_2 + t : sy_3) : (s : t) \in \mathbb{P}^1\}$$

intersecting $L = V(y_2)$ at the point $(y_0 : y_1 : 0 : y_3)$ for $(s : t) = (1 : -y_2)$. Identifying L with $\mathbb{P}^2$ by sending $(y_0 : y_1 : 0 : y_3) \in L$ to $(y_0 : y_1 : y_3) \in \mathbb{P}^2$ gives the claimed form of f.

From the form of the morphism $g : \mathbb{P}^1 \to \mathbb{P}^3$ given in the exercise, it's clear that a is not contained in the image $g(\mathbb{P}^1) = X$. Since a is the only point where f is not defined, the image $Y = f(X) = (f \circ g)(\mathbb{P}^1)$ is just the image of the morphism

$$f \circ g : \mathbb{P}^1 \to \mathbb{P}^2, (x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_1^3)\,.$$

Denote the coordinates of $\mathbb{P}^2$ by $(z_0 : z_1 : z_2)$ to avoid confusion. Then from the formula of $f \circ g$ we see that Y is contained in the vanishing locus $V(z_0^2 z_2 - z_1^3)$. In fact we claim that $f \circ g : \mathbb{P}^1 \to V(z_0^2 z_2 - z_1^3)$ is a bijection, so that $Y = V(z_0^2 z_2 - z_1^3)$. To prove this, let $U_2 = \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 : z_2 \neq 0\}$, then

$$(f \circ g)^{-1}(U_2) = \mathbb{P}^1 \setminus \{(1 : 0)\} \cong \mathbb{A}^1.$$

Then the unique preimage of $(z_0 : z_1 : 1) \in V(z_0^2 z_2 - z_1^3) \cap U_2$ is given by $(x_0 : 1) = (z_0/z_1 : 1)$ if $z_1 \neq 0$ and $(0 : 1)$ if $z_1 = 0$ (which forces $z_0 = 0$). On the other hand, the unique preimage of $V(z_0^2 z_2 - z_1^3) \cap V(z_2) = \{(1 : 0 : 0)\}$ under $f \circ g$ is given by $(1 : 0)$. This shows the claimed bijection $f \circ g : \mathbb{P}^1 \to V(z_0^2 z_2 - z_1^3)$.

b) We have already seen in class that $g : \mathbb{P}^1 \to \mathbb{P}^3$ is an isomorphism onto its image X. Thus $f : X \to Y$ is an isomorphism if and only if $f \circ g : \mathbb{P}^1 \to Y$ is an isomorphism. But by the analysis above, on the open subset $U_2 \cong \mathbb{A}^2 \subseteq \mathbb{P}^2$ the map $f \circ g$ is given by

$$\mathbb{A}^1 \to V(z_0^2 - z_1^3), x_0 \mapsto (x_0^3, x_0^2)$$

in affine coordinates (setting $x_1 = 1$ and $z_2 = 1$). We have seen in class that this map is not an isomorphism, and since an isomorphism must restrict to an isomorphism over any open subset of its image, this gives a contradiction.

Exercise 2.

a) For any $n, d \in \mathbb{N}_{>0}$, find explicit equations describing the image of the degree-d Veronese embedding

$$F : \mathbb{P}^n \to \mathbb{P}^N, (x_i)_{i=0,\dots,n} \mapsto (z_\alpha)_{\alpha \in \mathbb{N}^{n+1} : \sum \alpha_i = d} = \left(x^\alpha = \prod_{i=0}^n x_i^{\alpha_i} \right)_\alpha$$

of $\mathbb{P}^n$ in $\mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$.

b) Prove that every projective variety is isomorphic to the zero locus of quadratic polynomials in a projective space.

Solution.

a) Let $N_d = \{\alpha \in \mathbb{N}^{n+1} : \sum_i \alpha_i = d\}$ be the index set of the coordinates of $\mathbb{P}^N$. Then we claim that the image $X = F(\mathbb{P}^n) \subseteq \mathbb{P}^N$ is cut out by the ideal

$$J = \langle z_\alpha \cdot z_\beta - z_\gamma \cdot z_\delta : \alpha, \beta, \gamma, \delta \in N_d \text{ with } \alpha + \beta = \gamma + \delta \rangle$$

in the coordinate ring $K[z_\alpha : \alpha \in N_d]$ of $\mathbb{P}^N$. From the formula of F, it's clear that $X \subseteq V(J)$ since $x^\alpha \cdot x^\beta = x^\gamma \cdot x^\delta$ by the assumption $\alpha + \beta = \gamma + \delta$.

To show the other inclusion, note first that for $de_i = (0 : 0 : \ldots : 0 : d : 0 : \ldots 0) \in N_d$ we have that X is covered by the open sets $D(z_{de_i}) \subseteq \mathbb{P}^N$. Indeed, if all z_{de_i} vanish on $f(x)$ this forces $x_i^d = 0$ for all i, giving a contradiction. So let $z = (z_\alpha)_{\alpha \in N_d}$ be a point in $V(J)$ with $z_{de_i} \neq 0$, so that by rescaling we can assume $z_{de_i} = 1$. Then we claim that $z = F(x)$ for $x = (z_{(d-1)e_i + e_j})_{j=0,\dots,n}$.

We prove the equality $z_\alpha = F(x)_\alpha = x^\alpha$ for all $\alpha \in N_d$ by downward-induction on the i-th entry α_i of α. The cases $\alpha_i = d, d-1$ follow immediately from the formulas (using $z_{de_i} = x_i = 1$). On the other hand, assume the statement is proven for $\alpha \in N_d$ with $\alpha_i > d'$ and consider some α with $\alpha_i = d'$. Then setting $\beta = de_i$ we find $\gamma, \delta \in N_d$ with $\gamma_i, \delta_i > d'$ and $\alpha + \beta = \gamma + \delta$.[1] But then the defining equation of J forces $z_\alpha = z_\gamma \cdot z_\delta$. Since we already showed $z_\gamma = x^\gamma$, $z_\delta = x^\delta$ this implies

$$z_\alpha = x^{\gamma+\delta} = x^{\alpha+\beta} = x^\alpha \cdot \underbrace{x^\beta}_{=x_i^d = 1} = F(x)_\alpha$$

finishing the induction step.

[1] If $\alpha_j > 0$ for some index $j \neq i$ (which must exist) we can take $\gamma = \alpha + e_i - e_j$ and $\delta = (d-1)e_i + e_j$.

b) We have seen in class that any projective variety $Y \subseteq \mathbb{P}^n$ can be written as $Y = V(f_1, \ldots, f_m)$ for f_i homogeneous of the same degree d. Let $f_i = \sum_{\alpha \in N_d} c_{i,\alpha} x^\alpha$. Then we claim that

$$Y \cong \underbrace{V(\sum_{\alpha \in N_d} c_{i,\alpha} z_\alpha : i = 1, \ldots, n)}_{L} \cap F(\mathbb{P}^n) \subseteq \mathbb{P}^N .$$

Indeed, we know that $F : \mathbb{P}^n \to F(\mathbb{P}^n)$ is an isomorphism, and the pull-back of the defining equations of L are exactly the defining equations of Y. But L is a linear subspace of $\mathbb{P}^N$, and thus isomorphic to some $\mathbb{P}^{N'}$, and then the defining quadratic equations of $F(\mathbb{P}^n)$ found in the previous part of the exercise restrict to quadratic equations on L, cutting out Y.

Exercise 3. We denote the Plücker coordinates of the Grassmannian $G(2, 4)$ in $\mathbb{P}^5$ by x_{ij} for $1 \le i < j \le 4$.

a) Show that $G(2, 4) = V(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})$.

b) Let $L \subseteq \mathbb{P}^3$ be an arbitrary line. Show that the set of lines in $\mathbb{P}^3$ that intersect L, considered as a subset of $G(2, 4) \subseteq \mathbb{P}^5$, is the zero locus of a homogeneous linear polynomial.

How many lines in $\mathbb{P}^3$ would you expect to intersect four general given lines?

Solution.

a) In the lecture, we showed that the Grassmannian $G(2, 4)$ is cut out by the 3×3-minors of the matrix

$$M = \begin{pmatrix} x_{23} & -x_{13} & x_{12} & 0 \\ x_{24} & -x_{14} & 0 & x_{12} \\ x_{34} & 0 & -x_{14} & x_{13} \\ 0 & x_{34} & -x_{24} & x_{23} \end{pmatrix}$$

As an example, the upper left minor is given by

$$x_{23}x_{14}^2 + x_{13}x_{24}(-x_{14}) + x_{12}x_{34}x_{14} = x_{14}(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})$$

Thus we have

$$G(2, 4) \subseteq V(x_{14}) \cup V(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}) .$$

On the other hand, we know that $G(2, 4)$ is irreducible of dimension 4, and from Krull's principal ideal theorem one checks that the same is true for $V(x_{14})$ and $V(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})$. Since $G(2, 4)$ is not contained in $V(x_{14})$, it must be contained in $V(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})$, and for dimension reasons this containment must be an equality.

b) Assume L corresponds to the span $\mathrm{Lin}(a, b)$ of two vectors $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$. Let $L' = \mathrm{Lin}(c, d) \in G(2, 4)$ be another line. Then L, L' meet in a point if and only if $\mathrm{Lin}(a, b) \cap \mathrm{Lin}(c, d)$ has dimension at least 1 (and in this case, any one-dimensional sub-vectorspace contained in this intersection gives a point of $\mathbb{P}^3$ where L, L' meet.

Having such a positive-dimensional intersection is equivalent to the vectors a, b, c, d not being linearly independent, and thus equivalent to the vanishing of the determinant of the matrix

$$N = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} .$$

By expanding this determinant along the first two columns (applying Laplace expansion twice), we see that for fixed a_i, b_i, the determinant $\det(N)$ is a linear polynomial in the 2×2-minors of the last two columns. But these are exactly the Plücker coordinates of the line L'. Thus the condition $L \cap L' \neq \emptyset$ is described by a linear equation in the Plücker coordinates of L'.

Given four general lines $L_1, \ldots, L_4$, the set of lines in $G(2,4)$ intersecting each of them is described by a linear equation in the Plücker coordinates, i.e. a linear subspace $H_1, \ldots, H_4 \subseteq \mathbb{P}^5$ intersected with $G(2,4)$. The intersection $H_1 \cap \ldots \cap H_4$ is expected to be a line $\mathbb{P}^1 \subseteq \mathbb{P}^5$. By a), the Grassmannian is then cut out by a quadratic equation in the Plücker coordinates, which restricts to a quadratic equation on this $\mathbb{P}^1$. This has two zeros, counted with multiplicity, and by the assumption that all lines L_i were general, we expect that these two zeros are distinct, giving exactly two lines L, L' meeting all lines L_i.
Note: This argument can be made precise to show that there is a non-empty open subset U in $G(2,4)^4$ such that for a tuple $(L_1, L_2, L_3, L_4) \in U$, there are exactly two lines meeting all the lines L_i.

Exercise 4. Show that the following sets are projective varieties:

a) the incidence correspondence

$$\{(L, a) \in G(k,n) \times \mathbb{P}^{n-1} : L \subseteq \mathbb{P}^{n-1} \text{ a } (k-1)\text{-dimensional linear subspace and } a \in L\};$$

b) the join of two disjoint varieties $X, Y \subseteq \mathbb{P}^n$, i.e., the union of all lines in $\mathbb{P}^n$ intersecting both X and Y.

Solution.

a) Assume that $L = \mathrm{Lin}(b_1, \ldots, b_k)$ is the span of k vectors b_i, then the containment $a \in L$ is equivalent to the matrix

$$M = \begin{pmatrix} a_1 & b_{11} & \ldots & b_{k1} \\ a_2 & b_{12} & \ldots & b_{k2} \\ \vdots & & & \vdots \\ a_n & b_{1n} & \ldots & b_{kn} \end{pmatrix}$$

having rank exactly k (so that the first column a must be linearly dependent on the last k columns $b_1, \ldots, b_k$). Since the last k columns are already linearly independent (by the assumption that L is a point of $G(k,n)$), this is equivalent to all $(k+1) \times (k+1)$-minors of M vanishing. Expanding such a minor by the first column using the Laplace rule, we see that it is a linear combination of products of a_i with maximal minors of the matrix $B = (b_1, \ldots, b_k)$. These minors are the Plücker coordinates of L, whereas the a_i are the homogeneous coordinates of $\mathbb{P}^{n-1}$. On the standard open charts of $G(k,n)$ and $\mathbb{P}^{n-1}$ these become polynomial equations in the affine coordinates, so that the incidence correspondence is indeed a Zariski closed subset of $G(k,n)\times\mathbb{P}^{n-1}$. We have seen that both of these are projective varieties, and that the product of projective varieties is still projective. Thus the closed subset formed by the incidence correspondence is also projective.

b) Our first goal is to show that the variety

$$Z = \{L \in G(2,n+1) : L \text{ intersects both } X \text{ and } Y\} \subseteq G(2,n+1)$$

of all lines appearing in the definition of the join is projective. To see that, let $I = \{(L,a) : a \in L\}$ be the incidence correspondence from part a) applied to the case $G(2,n+1)$. Then I is projective and so the product

$$I^2 = I \times I = \{(L_1,a_1,L_2,a_2) : a_1 \in L_1, a_2 \in L_2\} \subseteq G(2,n+1)\times\mathbb{P}^n\times G(2,n+1)\times\mathbb{P}^n$$

is projective as well. We have two projection morphisms

$$\pi_1 : I^2 \to G(2,n+1)\times G(2,n+1) \text{ and } \pi_2 : I^2 \to \mathbb{P}^n \times \mathbb{P}^n$$

which remember the pairs (L_1,L_2) and (a_1,a_2) respectively. Let $\Delta \subseteq G(2,n+1)\times G(2,n+1)$ be the diagonal and $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^n$ be the product of X,Y, which are both closed subsets (here we use that $G(2,n)$ is a variety, and that the product of two closed embeddings is a closed embedding). Then we have that

$$\pi_1^{-1}(\Delta) \cap \pi_2^{-1}(X \times Y) = \{(L,a_1,L,a_2) : a_1 \in L \cap X, a_2 \in L \cap Y\}$$

is closed inside I^2 and thus projective as well. Our original variety Z at the beginning is the image of this projective variety under the projection to $G(2,n+1)$ (e.g. to the first factor). Since the image of a projective variety under a map to another variety is still projective, we conclude that Z is projective.

Finally, to obtain the join consider the diagram

$$G(2,n+1) \xleftarrow{p_1} I \xrightarrow{p_2} \mathbb{P}^n$$

with p_1, p_2 the projections to the two factors. Then the join $J(X,Y)$ is given by

$$J(X,Y) = p_2(p_1^{-1}(Z)) = p_2(\{(L,a) : L \text{ intersects both } X \text{ and } Y, a \in L\}.$$

Again since $Z \subseteq G(2,n+1)$ is closed, the closed subset $p_1^{-1}(Z) \subseteq I$ is projective and so its image $J(X,Y)$ under p_2 is projective as well.