Exercise Sheet 8

Exercise 1. Let $\widetilde{\mathbb{A}}^3$ be the blow-up of $\mathbb{A}^3$ at the line $L = V(x_1, x_2) \cong \mathbb{A}^1$. Show that its exceptional set is isomorphic to $\mathbb{A}^1 \times \mathbb{P}^1$. When do the strict transforms of two lines in $\mathbb{A}^3$ through L intersect in the blow-up? What is therefore the geometric meaning of the points in the exceptional set (analogously to the blow-up of a point, in which case the points of the exceptional set correspond to the directions through the blown-up point)?

Solution. The ideal of the line L is generated by x_1, x_2, so $\widetilde{\mathbb{A}}^3 = \mathrm{Bl}_{x_1,x_2}\mathbb{A}^3$ is contained in

$$\widetilde{\mathbb{A}}^3 \subseteq Y = V(x_1y_2 - x_2y_1) \subseteq \mathbb{A}^3 \times \mathbb{P}^1 \,.$$

As in [Gathmann, Example 9.15] we see that Y has one open patch

$$\{((x_1, x_2, x_3), (y_1 : y_2)) \in Y : y_1 \neq 0\} \cong \{((x_1, x_1y_2, x_3), (1 : y_2)) : (x_1, y_2, x_3) \in \mathbb{A}^3\} \cong \mathbb{A}^3$$

and similarly another patch $\mathbb{A}^3$ for $y_2 \neq 0$. Thus Y is irreducible of dimension 3 and thus for dimension reasons, the containment $\widetilde{\mathbb{A}}^3 \subseteq Y$ must be an equality.

To get the exceptional divisor, i.e. the preimage of $L = V(x_1, x_2)$ we set x_1, x_2 to zero and obtain

$$E = V(x_1, x_2, x_1y_2 - x_2y_1) = V(x_1, x_2) = \{(0, 0)\} \times \mathbb{A}^1 \times \mathbb{P}^1 \subseteq \mathbb{A}^3 \times \mathbb{P}^1 \,.$$

Let $L_1, L_2 \subseteq \mathbb{A}^3$ be lines through L. If their strict transforms $\widetilde{L}_1, \widetilde{L}_2$ meet, then also L_1, L_2 have to meet (since the map $\widetilde{\mathbb{A}}^3 \to \mathbb{A}^3$ sends $\widetilde{L}_i$ to L_i. So assume that L_1, L_2 go through a point in L. By translation along the third coordinate, let's assume they go through the origin $(0, 0, 0) \in L$, and so they are given as

$$L_i = \{s_i(a_{i,1}, a_{i,2}, a_{i,3}) : s_i \in \mathbb{A}^1\} \,.$$

Since none of the two lines are equal to L (because in this case their strict transform is empty), we have $(a_{i,1}, a_{i,2}) \neq 0$. Then for $s_i \neq 0$ the corresponding point of $L_i \setminus \{0\} \subseteq Y$ is given by

$$((s_ia_{i,1}, s_ia_{i,2}, s_ia_{i,3}), (a_{i,1} : a_{i,2})) \in Y \subseteq \mathbb{A}^3 \times \mathbb{P}^1 \,.$$

Indeed, the equation $x_1y_2 - x_2y_1$ exactly forces $(y_1 : y_2) = (s_ia_{i,1} : s_ia_{i,2}) = (a_{i,1} : a_{i,2}) \in \mathbb{P}^1$. Taking the limit $s_i = 0$ we obtain the point $((0, 0, 0) : (a_{i,1} : a_{i,2})) \in \widetilde{L}_i$. Thus the two strict transforms meet if and only if $(a_{1,1} : a_{1,2}) = (a_{2,1} : a_{2,2}) \in \mathbb{P}^1$.

As a geometric interpretation: the lines on the exceptional set parameterize the choice of a point on L together with a normal direction in $K^3/\{(0, 0)\} \times K \cong K^2$ up to scaling. This explains why $E \cong \mathbb{A}^1 \times \mathbb{P}^1 = L \times \mathbb{P}^1$.

Exercise 2. Show that any irreducible quadric hypersurface $Q \subseteq \mathbb{P}^n$ over a field of characteristic not equal to 2 is birational to $\mathbb{P}^{n-1}$. Can you give an example of some Q which is not isomorphic to $\mathbb{P}^{n-1}$?

Solution. The basic idea to get the birational map $Q \dashrightarrow \mathbb{P}^{n-1}$ is to project from a point $p_0 \in Q$ to some hyperplane $H \cong \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$.

To make our life easier, we can use a projective automorphism of $\mathbb{P}^n$ to move some point of Q to $p_0 = (1 : 0 : \ldots : 0)$, so without loss of generality we can assume that the above p_0 is contained in Q. Then we project to the hyperplane $H = V(x_0)$. The resulting morphism

$$f : Q \setminus \{p_0\} \to \mathbb{P}^{n-1}, (x_0 : x_1 : \ldots, x_n) \mapsto (x_1 : \ldots, x_n)$$

is defined away from p_0. To write down the inverse, let

$$F = a_0x_0^2 + a_1x_0 + a_2 \in K[x_0, x_1, \ldots, x_n]_2$$

be the quadratic equation cutting out $Q = V(F)$. Here

$$a_0 \in K, a_1 \in K[x_1, \ldots, x_n]_1 \text{ and } a_2 \in K[x_1, \ldots, x_n]_2$$

are the coefficients, seeing F as a polynomial in x_0 over $K[x_1, \ldots, x_n]$. The containment $p_0 \in Q$ forces $F(1, 0, \ldots, 0) = 0$ which implies $a_0 = 0$. Then we distinguish two cases:

Case 1: $a_1 = 0$

Then the equation of F does not depend on x_0 at all. By induction on n we know that $V(a_2) \subseteq \mathbb{P}^{n-1}$ is birational to $\mathbb{P}^{n-2}$, say by some rational map $g : \mathbb{P}^{n-2} \dashrightarrow V(a_2)$. Then one can check that

$$\mathbb{P}^1 \times \mathbb{P}^{n-2} \dashrightarrow Q, ((s : t), (y_1 : y_2 : \ldots : y_n)) \mapsto (s_0 : ty_1 : \ldots : ty_n)$$

is a birational map. Since $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ is birational to $\mathbb{P}^{n-1}$, this finishes the proof.

Case 2: $a_1 \neq 0$

Then on the locus $U = \mathbb{P}^{n-1} \setminus V(a_1) \subseteq \mathbb{P}^{n-1}$ the inverse of the projection f is given by

$$U \to Q, (x_1, \ldots, x_n) \mapsto \left(-\frac{a_2(x)}{a_1(x)} : x_1 : \ldots : x_n\right) = (-a_2(x) : x_1a_1(x) : \ldots : x_na_1(x)) .$$

All the components of this map are homogeneous polynomials of degree 2, which don't vanish simultaneously (since $(x_1, \ldots, x_n) \neq 0$ and $a_1(x) \neq 0$ on its domain). Hence by [Gathmann, Lemma 7.4] this indeed defines a morphism, and by a short calculation it is the inverse of f. Hence f is birational, as claimed.

For an example of Q which is not isomorphic to $\mathbb{P}^{n-1}$ take $Q = V(x_0x_3 - x_1x_2) \subseteq \mathbb{P}^3$. Then we have seen in [Gathmann, Example 7.11] that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, and on Sheet 6, Exercise 1, we proved that $\mathbb{P}^1 \times \mathbb{P}^1 \ncong \mathbb{P}^2$.

Exercise 3. Let $X \subseteq \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subseteq X$ be irreducible, closed subsets, none contained in the other. Moreover, let $\widetilde{X}$ be the blow-up of X at the ideal $I(Y_1) + I(Y_2)$. Show that the strict transforms of Y_1 and Y_2 in $\widetilde{X}$ are disjoint.

Solution. Let $I(Y_1) = \langle f_1, \ldots, f_r\rangle$ and $I(Y_2) = \langle f_{r+1}, \ldots, f_{r+s}\rangle$, then of course

$$I(Y_1) + I(Y_2) = \langle f_1, \ldots, f_{r+s}\rangle \trianglelefteq K[x_1, \ldots, x_n] .$$

Thus we can calculate $\widetilde{X} = \mathrm{Bl}_{f_1,\ldots,f_{r+s}}X$ as the blow-up at the union of these generator sets. Then we find

$$\widetilde{X} \subseteq Z = \{((x_1, \ldots, x_n), (y_1, \ldots, y_{r+s})) \in \mathbb{A}^n \times \mathbb{P}^{r+s-1} : y_if_j(x) = y_jf_i(x) \forall i, j\} .$$

Claim: $\widetilde{Y}_1 \subseteq V(y_1, \dots, y_r) \subseteq Z$ and $\widetilde{Y}_2 \subseteq V(y_{r+1}, \dots, y_{r+s}) \subseteq Z$.
Assuming the claim, we have

$$\widetilde{Y}_1 \cap \widetilde{Y}_2 \subseteq V(y_1, \dots, y_{r+s}) = \emptyset\,,$$

since not all coordinates of the point $y \in \mathbb{P}^{r+s-1}$ can vanish simultaneously. Thus $\widetilde{Y}_1$ and $\widetilde{Y}_2$ are indeed disjoint.
Proof of claim: We prove the statement for $\widetilde{Y}_1$, with $\widetilde{Y}_2$ working similarly. By the assumption that Y_1 is not contained in Y_2, we have that $U = Y_1 \setminus Y_2 \subseteq Y_1$ is open, and thus dense since Y_1 is irreducible. Let $i = 1, \dots, r$, then we claim that y_i vanishes at any point $x \in U$ (and thus also on the closure $\widetilde{Y}_1$ of U in $\widetilde{Y}_1$, finishing the proof of the claim). Since $x \in U$ is disjoint from $Y_2 = V(f_{r+1}, \dots f_{r+s})$ we find an index $j \in \{r+1, \dots, r+s\}$ such that $f_j(x) \neq 0$. But then the equation

$$y_i \underbrace{f_j(x)}_{\neq 0} = y_j \underbrace{f_i(x)}_{=0 \text{ as } x \in Y_1}$$

implies $y_i = 0$ as claimed.

Exercise 4. Let $J \trianglelefteq \mathbb{K}[x_1, \dots, x_n]$ be an ideal, and assume that the corresponding affine variety $X = V(J) \subseteq \mathbb{A}^n$ contains the origin. Consider the blow-up $\widetilde{X} \subseteq \widetilde{\mathbb{A}}^n \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ at $x_1, \dots, x_n$, and denote the homogeneous coordinates of $\mathbb{P}^{n-1}$ by $y_1, \dots, y_n$.

a) We know already that $\widetilde{\mathbb{A}}^n$ can be covered by affine spaces, with one coordinate patch being

$$i : U = \mathbb{A}^n \to \widetilde{\mathbb{A}}^n \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1},$$
$$(x_1, y_2, \dots, y_n) \mapsto ((x_1, x_1 y_2, \dots, x_1 y_n), (1 : y_2 : \cdots : y_n)).$$

Prove that on this coordinate patch the blow-up $\widetilde{X}$ is given as the zero locus of the polynomials

$$\frac{f(x_1, x_1 y_2, \dots, x_1 y_n)}{x_1^{\text{min deg } f}}$$

for all non-zero $f \in J$, where min deg f denotes the smallest degree of a monomial in f.
Hint: You can use without proof the following variant of [Gathmann, Exercise 2.23]:

For $I, J \trianglelefteq K[x_1, \dots, x_n]$ one has $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$ where

$$(I : J^\infty) = \{f \in K[x_1, \dots, x_n] : \exists m \in \mathbb{N}, g \in J^m \text{ with } fg \in I\}\,.$$

b) Show that the exceptional set of the blow-up $\widetilde{X}$ is

$$V_p\left(f^{\text{in}}(y) : f \in J\right) \subseteq \mathbb{P}^{n-1} \cong \{0\} \times \mathbb{P}^{n-1},$$

where f^{in} is the initial term of f, i.e. the sum of all monomials in f of smallest degree. Consequently, the tangent cone of X at the origin is

$$C_0 X = V_a(f^{\text{in}} : f \in J) \subseteq \mathbb{A}^n.$$

Solution.

a) By definition, the blow-up $\widetilde{X}$ is obtained by taking the closure of $X \setminus \{0\}$ inside $\mathbb{A}^n \times \mathbb{P}^{n-1}$, which is automatically contained in $\widetilde{\mathbb{A}}^n$ as seen in the lecture. Taking this closure and intersecting with the open patch $U = \mathbb{A}^n \subseteq \widetilde{\mathbb{A}}^n$ mentioned above, is equivalent (by basic topology) to first intersecting with U and then taking the closure.

By the map $i : U = \mathbb{A}^n \to \widetilde{\mathbb{A}}^n$, we have

$$i^{-1}V(f(x_1, \dots, x_n)) = V(f(x_1, x_1y_2, \dots, x_1y_n)),$$

so

$$\begin{aligned} i^{-1}(X \setminus \{0\}) &= i^{-1}(V(f : 0 \neq f \in J) \setminus V(x_1, \dots, x_n)) \\ &= V(f(x_1, x_1y_2, \dots, x_1y_n) : 0 \neq f \in J) \setminus \underbrace{V(x_1, x_1y_2, \dots, x_1y_n)}_{=V(x_1)} . \end{aligned}$$

To take the Zariski closure $\overline{i^{-1}(X \setminus \{0\})}$, we just apply the hint given above, and we see that this closure is cut out by the ideal

$$(\langle f(x_1, x_1y_2, \dots, x_1y_n) : 0 \neq f \in J\rangle : \langle x_1\rangle^\infty) = \langle \frac{f(x_1, x_1y_2, \dots, x_1y_n)}{x_1^{\text{min deg } f}} : 0 \neq f \in J\rangle .$$

b) We check the equality on the open subset U above. To get the exceptional set, we impose the additional condition $x_1 = 0$. Given $0 \neq f \in J$ with minimal degree d and total degree e we write its homogeneous decomposition as $f = f^{\text{in}} + f_{d+1} + \dots + f_e$. Then we have

$$f(x_1, x_1y_2, \dots, x_1y_n) = x_1^d f^{\text{in}}(1, y_2, \dots, y_n) + x_1^{d+1} f_{d+1}(1, y_2, \dots, y_n) + \dots + x_1^e f_e(1, y_2, \dots, y_n) .$$

Dividing by $x_1^{\text{min deg} f} = x_1^d$ and setting $x_1 = 0$, all the terms except the first vanish, and we have

$$\frac{f(x_1, x_1y_2, \dots, x_1y_n)}{x_1^{\text{min deg } f}}|_{x_1=0} = f^{\text{in}}(1, y_2, \dots, y_n) .$$

This is exactly the initial term of f in the affine coordinates $U_0 \subseteq \mathbb{P}^{n-1}$, which proves the first statement.

The second is then just an application of this result to the definition of the tangent cone at the origin.