Exercise Sheet 9

Exercise 1. Prove the projective Jacobi criterion:
Let $X \subseteq \mathbb{P}^n$ be a projective variety with ideal $I(X) = \langle f_1, \ldots, f_r \rangle$, and let $a \in X$. Then X is smooth at a if and only if the rank of the $r \times (n+1)$ Jacobi matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{i,j}$$

is at least $n - \operatorname{codim}_X\{a\}$.
Hint: Show and use that

$$\sum_{i=0}^{n} x_i \cdot \frac{\partial f}{\partial x_i} = df$$

for every homogeneous polynomial $f \in K[x_0, \ldots, x_n]$ of degree d.

Solution. We first prove the hint: since both sides are linear in f, we can reduce to the case where f is a monomial $f = x^{\mathbf{e}} = x_0^{e_0} \cdots x_n^{e_n}$, for which we calculate:

$$\sum_{i=0}^{n} x_i \cdot \frac{\partial f}{\partial x_i} = \sum_{i=0}^{n} e_i \cdot \underbrace{x_i \cdot x_0^{e_0} \cdots x_i^{e_i - 1} \cdots x_n^{e_n}}_{= x^{\mathbf{e}}} = x^{\mathbf{e}} \cdot \underbrace{\sum_{i=0}^{n} e_i}_{=d} = f \cdot d\,.$$

To check the smoothness of X at a let us assume that e.g. $a_0 \neq 0$. Then we can reduce the problem to the affine Jacobi criterion on $U_0 = \{x \in \mathbb{P}^n : x_0 \neq 0\}$. Here X is cut out by the dehomogenizations $f_j^i(x_1, \ldots, x_n) = f_j(1, x_1, \ldots, x_n)$ of the generators of $I(X)$. Picking a representative of $a \in X \subseteq \mathbb{P}^n$ with $a_0 = 1$ we see that columns $j = 1, \ldots, n$ of the matrix J are exactly the affine Jacobian matrix J_a of $X \cap U_0$. The 0-th column containing the derivatives $\partial f_i / \partial x_0(a)$ seems redundant and like it could potentially lead to a bigger rank. However, using the hint for f_i (which is homogeneous of some degree d_i) and evaluating at a, we see that

$$\sum_{j=0}^{n} a_j \cdot \frac{\partial f_i}{\partial x_j}(a) = d_i f_i(a) = 0\,.$$

This shows that taking the linear combination $\sum_j a_j c_j$ of the columns c_j of J gives zero. Since the coefficient a_0 of the zeroth column c_0 is nonzero, it is linearly dependent from the others. Thus the rank of J (at a) equals the rank of the matrix J_a from the affine Jacobi criterion [Gathmann, Proposition 10.11]. Thus the result follows from that criterion.

Exercise 2. For $k \in \mathbb{N}_{>0}$ let X_k be the complex singular affine curve

$$X_k := V(x_2^2 - x_1^{2k+1}) \subseteq \mathbb{A}^2_{\mathbb{C}}\,,$$

and denote by $\widetilde{X}_k \subseteq \widetilde{\mathbb{A}}^2$ the blow-ups of X_k and $\mathbb{A}^2$ at the origin, respectively.

a) Use suitable coordinates on $\widetilde{\mathbb{A}}^2$ to determine all k for which $\widetilde{X}_k$ is smooth.

b) Show that X_k is not isomorphic to X_l if $k \neq l$.

Hint: Follow the idea of [Gathmann, Example 10.16].

Solution.

a) Following [Gathmann, Example 10.16] as suggested, we can use coordinates $((x_1, x_2), (y_1 : y_2))$ on $\widetilde{\mathbb{A}}^2_{\mathbb{C}}$, and we look at the chart $y_1 \neq 0$ with coordinates x_1, y_2. Applying [Gathmann, Exercise 9.22 (a)] the blow-up in this chart is then given by

$$\widetilde{X}_k = V\left(\frac{(x_1 y_2)^2 - x_1^{2k+1}}{x_1^2}\right) = V(\underbrace{y_2^2 - x_1^{2(k-1)+1}}_{=:g}).$$

The exceptional set, obtained by setting $x_1 = 0$, then contains the single point $((0,0),(1:0))$. The affine Jacobi matrix (for the coordinates x_1, y_2 around that point) is given by

$$(\partial g/\partial x_1 \;\; \partial g/\partial y_2)(0,0) = (-(2(k-1)+1)x_1^{2k-2} \;\;\; 2y_2)(0,0) = \begin{cases} (1\ 0) & \text{for } k=1 \\ (0\ 0) & \text{for } k>1\,. \end{cases}$$

By the affine Jacobi criterion, we see that $\widetilde{X}_k$ is smooth at $((0,0),(1:0))$ if and only if $k=1$ (in which case we are looking at the example treated in [Gathmann, Example 10.16], which was seen to be smooth everywhere). Thus $\widetilde{X}_k$ is smooth if and only $k=1$.

b) All the varieties X_k are have a unique singular point, sitting at the origin. Moreover, as shown in part a), the blow-up $\widetilde{X}_k$ at this singular point has an exceptional set consisting of a single point, which is singular if and only if $k=1$. Even better, as the form of g above shows, this singular point has a neighborhood isomorphic to the curve X_{k-1}. Thus we can iterate the blow-up procedure, always blowing up the remaining singular point. Then we can uniquely reconstruct the index k of X_k as the number of times we have to blow up before we obtain a smooth curve. Since this number is uniquely determined like this, any isomorphism $X_k \cong X_l$ would have to imply $k = l$.

Exercise 3. Let $n \geq 2$. Prove:

a) Every smooth hypersurface in $\mathbb{P}^n$ is irreducible.

b) A general hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ is smooth (and thus by a) irreducible). More precisely, for a given $d \in \mathbb{N}_{>0}$ the vector space $\mathbb{C}[x_0, \ldots, x_n]_d$ has dimension

$$\binom{n+d}{n},$$

and so the space of all homogeneous degree-d polynomials in $x_0, \ldots, x_n$ modulo scalars can be identified with the projective space $\mathbb{P}_{\mathbb{C}}^{(\binom{n+d}{n}-1)}$. Show that the subset of this projective space of all (classes of) polynomials f such that f is irreducible and $V_p(f)$ is smooth is dense and open.

Solution.

a) Assume X was a smooth reducible hypersurface, with at least two different irreducible components X_1, X_2. By definition of a hypersurface, both of them have dimension $n-1$. We claim that they must intersect at some point $a \in \mathbb{P}^n$. Indeed, their dimensions satisfy

$$\dim X_i + \dim X_j = 2n - 2 \geq n \text{ as } n \geq 2$$

and so by [Gathmann, Exercise 6.31] their intersection is non-empty. But we have seen ([Gathmann, Remark 10.10]) that smooth varieties are locally irreducible, giving a contradiction to X being smooth at $a \in X_1 \cap X_2$.

b) Let $\mathrm{Poly}_{d,n} = \mathbb{P}_{\mathbb{C}}^{(\binom{n+d}{n}-1)}$ be the space of polynomials $0 \neq f \in \mathbb{C}[x_0, \ldots, x_n]_d$ up to scaling. We have the universal hypersurface

$$\mathcal{X}_{d,n} = \{([f], a) \in \mathrm{Poly}_{d,n} \times \mathbb{P}^n : f(a) = 0\} \xrightarrow{\pi} \mathrm{Poly}_{d,n}.$$

Inside $\mathcal{X}_{d,n}$ consider the closed subset $\mathcal{X}_{d,n}^{\mathrm{sing}}$ of pairs $([f], a)$ cut out by the condition that the Jacobi matrix J from Exercise 1 vanishes. We claim that the locus of $[f]$ such that f is reducible or $V_p(f)$ is singular is given by $\pi(\mathcal{X}_{d,n}^{\mathrm{sing}})$. Since π is a closed map, this would prove that this locus is closed. Moreover, its open complement contains the equation $[f] = [x_0^d + \ldots + x_n^d]$ of the Fermat hypersurface by [Gathmann Example 10.21] and thus is non-empty (here we use that we work over $\mathbb{C}$). Since $\mathrm{Poly}_{d,n}$ is irreducible, this would finish the proof.

To show the claim, first assume that $0 \neq f$ is not irreducible. Then either it contains multiple irreducible factors (in which case $V_p(f)$ is not smooth by a)) or it is a power of a lower-degree polynomial. In both cases (either by Exercise 1 or by a direct calculation), the Jacobi matrix vanishes. Here we use that $X = V_p(f)$ is of pure dimension $n-1$, so that $\mathrm{codim}_X\{a\} = n-1$ for all $a \in X$.

Likewise, if f is irreducible but $V_p(f)$ is not smooth at some point a, we have that $([f], a) \in \mathcal{X}_{d,n}^{\mathrm{sing}}$. On the other hand, if f is irreducible and $V_p(f)$ is smooth, then no point $([f], a)$ in $\pi^{-1}([f])$ can be contained in $\mathcal{X}_{d,n}^{\mathrm{sing}}$. This finishes the claim and thus the proof.

Exercise 4. Assume that the characteristic of K is not equal to 2, and let f be a homogeneous polynomial in $K[x_0, x_1, x_2]$ whose partial derivatives $\frac{\partial f}{\partial x_i}$ for $i = 0, 1, 2$ do not vanish simultaneously at any point of $X = V_p(f) \subseteq \mathbb{P}^2$. Then the image of the morphism

$$F : X \to \mathbb{P}^2, a \mapsto \left(\frac{\partial f}{\partial x_0}(a) : \frac{\partial f}{\partial x_1}(a) : \frac{\partial f}{\partial x_2}(a)\right)$$

is called the dual curve to X.

a) Find a geometric description of F. What does it mean geometrically if $F(a) = F(b)$ for two distinct points a, b in X?

b) If X is a conic (i.e., an irreducible curve of degree 2), prove that its dual $F(X)$ is also a conic.

c) For any five lines in $\mathbb{P}^2$ in general position, show that there is a unique conic in $\mathbb{P}^2$ that is tangent to all of them.
Hint: You can use without proof that the dual curve of the dual curve is again the original curve.

Solution.

a) We see the target $\mathbb{P}^2$ of F as the space of lines inside $\mathbb{P}^2$ by identifying $(b_0 : b_1 : b_2) \in \mathbb{P}^2$ with the line

$$L_b = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : b_0x_0 + b_1x_1 + b_2x_2 = 0\}.$$

Then we claim that the line $L_{F(a)}$ is precisely the tangent line to X at a. Indeed, it follows from the hint in Exercise 1 above that $a \in L_{F(a)}$ (by a short calculation), and in affine coordinates one can verify that the tangent space is precisely cut out by the condition from $L_{F(a)}$. Thus $F(a) = F(b)$ if and only if the tangent lines to X at a, b coincide.

b) For X a conic, all partial derivatives $\partial f/\partial x_i$ are linear polynomials, so the map above is linear. We claim that it is in fact the restriction of a projective automorphism, i.e. the corresponding matrix is invertible. If it wasn't, then there would be a point $A \in \mathbb{P}^2$ where all partial derivatives $\partial f/\partial x_i(A)$ vanish. But again by the hint from Exercise 1 we would then also have $d \cdot f(A) = 2 \cdot f(A) = 0$, so $A \in X$, giving a contradiction to the assumption that not all partial derivatives of f vanish at any point of X. Thus the matrix is a projective automorphism, and thus sends a conic to a conic.

c) The condition that the conic X is tangent to five lines $L_1, \ldots, L_5$ is equivalent to saying that $X^\vee = F(X)$ passes through those lines (seen as elements of $\mathbb{P}^2$). By [Gathmann, Exercise 7.30] there is a unique conic $X^\vee$ passing through those five general points. But then $X = F(X^\vee)$ is again a conic (by part b) and satisfies $F(X) = X^\vee$ by the hint. Moreover, it is the unique such conic since the assignment $X \mapsto X^\vee$ is bijective.