# Exercise Sheet 2

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**Exercise 1** (Transitive Group Actions)**.** Let $G$ be a topological group, $X$ a topological space and $\mu : G \times X \to X$ a continuous transitive group action, i.e. for any two $x, y \in X$ there is $g \in G$ such that $\mu(g, x) = g \cdot x = y$.

a) Show that if $G$ is compact then $X$ is compact.

b) Show that if $G$ is connected then $X$ is connected.

**Exercise 2** (Examples of Haar Measures)**.**

a) Let us consider the *three-dimensional Heisenberg group* $H = \mathbb{R} \rtimes_\eta \mathbb{R}^2$, where $\eta : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is defined by

$$\eta(x)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z + xy \end{pmatrix},$$

for all $x, y, z \in \mathbb{R}$. Thus the group operation is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and it is easy to see that it can be identified with the matrix group

$$H \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

Verify that the Lebesgue measure is the Haar measure of $\mathbb{R} \rtimes_\eta \mathbb{R}^2$ and that the group is unimodular.

b) Let

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}.$$

Show that $\frac{da}{a^2}\, db$ is the left Haar measure and $da\, db$ is the right Haar measure. In particular, $P$ is *not* unimodular.

c) Let $G := \mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ denote the group of invertible matrices over $\mathbb{R}$. Let $\lambda_{n^2}$ denote the Lebesgue measure on $\mathbb{R}^{n^2}$. Prove that

$$\mathrm{d}m(x) := |\det x|^{-n}\, \mathrm{d}\lambda_{n^2}(x)$$

defines a bi-invariant (i.e. left- and right-invariant) Haar measure on $G$.

d) Let $G = \mathrm{SL}_n(\mathbb{R})$ denote the group of matrices of determinant 1 in $\mathbb{R}^{n\times n}$. For a Borel subset $B \subseteq \mathrm{SL}_n(\mathbb{R})$ define

$$m(B) := \lambda_{n^2}\big(\{tg; g \in B, t \in [0,1]\}\big).$$

Show that $m$ is a well-defined bi-invariant Haar measure on $\mathrm{SL}_n(\mathbb{R})$.

e) Let $G$ denote the $ax+b$ group defined as

$$G = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix}; a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$$

Note that every element in $G$ can be written in a unique fashion as a product of the form:

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$, which yields a coordinate system $\mathbb{R}^\times \times \mathbb{R} \leftrightarrow G$. Prove that

$$\mathrm{d}m(\alpha, \beta) = \frac{1}{|\alpha|}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

defines a left Haar measure on $G$. Calculate $\Delta_G(\alpha, \beta)$ for $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$.

**Exercise 3** ($\mathrm{Aut}(\mathbb{R}^n, +) \cong \mathrm{GL}(n, \mathbb{R})$)**.** For a topological group $G$, we denote by $\mathrm{Aut}(G)$ the group of bijective, continuous homomorphisms of $G$ with continuous inverse. Consider the locally compact Hausdorff group $G = (\mathbb{R}^n, +)$ where $n \in \mathbb{N}_0$.

a) Show that $\mathrm{Aut}(G)$, i.e. the group of bijective homomorphisms which are homeomorphisms as well, is given by $\mathrm{GL}_n(\mathbb{R})$.

b) Show that $\mathrm{mod} : \mathrm{Aut}(G) \to \mathbb{R}_{>0}$ is given by $\alpha \mapsto |\mathrm{det}\alpha|$.

**Remark.** By the definition given in the lecture $\mathrm{mod}(\alpha)$ is the unique positive real number such that $m(\alpha \cdot f) = \mathrm{mod}(\alpha) m(f)$ for all $f \in C_c(G)$, $m$ left-Haar measure on $G$.
This definitions may differ by an inverse from other definitions in the literature.

c) Prove that there exists a discontinuous, bijective homomorphism from the additive group $(\mathbb{R}, +)$ to itself.