Exercise Sheet 3

Exercise 1. For each of the following locally compact Hausdorff groups, give an example of a lattice or prove that it does not admit a lattice.

a) The free group on 2 generators with the discrete topology.

b) $G = \mathrm{SO}(n, \mathbb{R})$.

c) $G = (\mathbb{R}_{>0}, \cdot)$.

d) $G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$, see exercise 2b) on Sheet 2.

e) The Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$

see exercise 2a) on Sheet 2.

Exercise 2 (Regular Subgroups are closed)**.** Let G be a Lie group, $H \leq G$ a subgroup that is also a regular submanifold. Prove that H is a closed subgroup of G.

Exercise 3 (The Matrix Lie Group $O(p, q)$)**.** Let $p, q \in \mathbb{N}$ and $n = p + q$. We define the (indefinite) symmetric bilinear form $\langle \cdot, \cdot \rangle_{p,q}$ of signature (p, q) on $\mathbb{R}^n$ to be

$$\langle v, w \rangle_{p,q} := v_1 w_1 + \cdots + v_p w_p - v_{p+1} w_{p+1} - \cdots - v_{p+q} w_{p+q}$$

for all $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. Show that

$$O(p, q) := \{ A \in \mathrm{GL}(n, \mathbb{R}) : \langle Av, Aw \rangle_{p,q} = \langle v, w \rangle_{p,q} \quad \forall v, w \in \mathbb{R}^n \}.$$

is a Lie group using the inverse function theorem/constant rank theorem. What is its dimension?

Exercise 4. Let M be a smooth n-dimensional manifold and $p \in M$. Show that if (U, φ) is any chart at p with $\varphi(p) = 0$, then the map

$$\mathbb{R}^n \to T_p M, \quad v \mapsto (f \mapsto D_0(f \circ \varphi^{-1})(v))$$

is a vector space isomorphism.