Exercise Sheet 4

Exercise 1. We consider the determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$. Show that its differential at the identity matrix I is the trace function

$$D_I \det = \mathrm{tr}.$$

This calculation was used in the lecture to show that the Lie algebra of $\mathrm{SL}(n, \mathbb{R})$ is the set of traceless matrices.

Exercise 2 (Some Lie Algebras)**.** (a) Let M, N be smooth manifolds and let $f : M \to N$ be a smooth map of constant rank r. By the constant rank theorem we know that the level set $L = f^{-1}(q)$ is a regular submanifold of M of dimension $\dim M - r$ for every $q \in N$. Show that one may canonically identify

$$T_p L \cong \ker d_p f$$

for every $p \in L = f^{-1}(q)$.

Hint: Describe elements in $T_p L$ as $\gamma'(0)$ for a smooth path $\gamma \colon (-\varepsilon, \varepsilon) \to M$.

(b) Use part a) to compute the Lie algebras of the following Lie groups: $\mathrm{O}(n, \mathbb{R})$, $\mathrm{SO}(n, \mathbb{R})$, $\mathrm{O}(p, q)$, $B(n)$ the group of real invertible upper triangular matrices and $N(n)$ the subgroup of $B(n)$ with only ones on the diagonal.

Exercise 3 (One- and two-dimensional Lie Algebras)**.** Classify the one- and two-dimensional real Lie algebras up to Lie algebra isomorphism and realize them as Lie subalgebras of some $\mathfrak{gl}_n \mathbb{R} = \mathfrak{gl}(\mathbb{R}^n)$.

Hint: In dimension two one can show that if the Lie algebra is non-abelian then there is a basis X, Y such that $[X, Y] = Y$.

Exercise 4. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Show that the Lie algebra of $G \times H$ can be identified with $\mathfrak{g} \times \mathfrak{h}$ with the bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2]_{\mathfrak{g}}, [y_1, y_2]_{\mathfrak{h}}).$$

Exercise 5 (Surjectivity of the Matrix Exponential)**.** Let $\mathrm{Exp} : \mathfrak{gl}(n, \mathbb{R}) \cong \mathbb{R}^{n \times n} \to \mathrm{GL}(n, \mathbb{R})$ be the matrix exponential map given by the power series

$$\mathrm{Exp}(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Consider the Lie subgroup of upper triangular matrices with 1's on the diagonal $N(n) < \mathrm{GL}(n, \mathbb{R})$ with its Lie algebra $\mathfrak{n}(n) < \mathfrak{gl}(n, \mathbb{R})$ of strictly upper triangular matrices; cf. Exercise 1.

Show that $\mathrm{Exp}|_{\mathfrak{n}(n)} : \mathfrak{n}(n) \to N(n)$ is surjective.

Hint: Consider the partially defined matrix logarithm $\mathrm{Log} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$\mathrm{Log}(I + A) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n}.$$

Try to give answers to the following questions and then conclude:

What is its radius of convergence r about I? Why is it a right-inverse of Exp on the ball $B_r(I)$ of radius r about I? Why is there no problem for matrices that are in $N(n)$ but not in $B_r(I)$?

In order to answer the last question prove that $A^n = 0$ for all $A \in \mathfrak{n}(n)$.