Exercise Sheet 5

Exercise 1 (Discrete Subgroups of $\mathbb{R}^n$)**.** Let $D < \mathbb{R}^n$ be a discrete subgroup. Show that there are $x_1, \ldots, x_k \in D$ such that

- $x_1, \ldots, x_k$ are linearly independent over $\mathbb{R}$, and
- $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$, i.e. $x_1, \ldots, x_k$ generate D as a $\mathbb{Z}$-submodule of $\mathbb{R}^n$.

Exercise 2. Show that every connected abelian Lie group G is isomorphic as a Lie group to $\mathbb{T}^a \times \mathbb{R}^{n-a}$ for some $a \in \{0, \ldots, n\}$, where $n = \dim G$ and $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$.

Exercise 3 (Easy Direction of Frobenius' Theorem)**.** Let M be a smooth manifold and let $\mathcal{D}$ be a distribution on M. Show that $\mathcal{D}$ is involutive if it is completely integrable.

Exercise 4. (a) Show that every continuous group homomorphism from $\mathbb{R}$ to a Lie group is smooth.

(b) Show that every continuous group homomorphism between two Lie groups is smooth.

In fact, one can prove that every measurable homomorphism between two locally compact topological groups is continuous, hence every measurable homomorphism between two Lie groups is smooth.

Exercise 5. (Corollary 3.93(2)) Show that if two simply connected, connected Lie groups G_1, G_2 have isomorphic Lie algebra, then they are isomorphic.