Exercise Sheet 6

Exercise 1 (The adjoint representatio is smooth)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}(g)$ is smooth, where $\mathrm{Ad}(g) := D_e(\mathrm{int}(g))$.

Hint: Consider the map $\mathrm{int}(g)\colon G \to G$, $x \mapsto gxg^{-1}$. Use that $\exp_G$ is a local diffeomorphism to conclude that Ad is smooth near e. Then use left translation to show that Ad is smooth everywhere.

Exercise 2 ($\mathrm{Z}(G) = \mathrm{Ker}(\mathrm{Ad})$)**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Use the fundamental relation that

$$g \exp(tX) g^{-1} = \exp(t \,\mathrm{Ad}_g(X))$$

for all $g \in G, t \in \mathbb{R}$ and $X \in \mathfrak{g}$ to prove the following.

(1) If G is connected, then the center $\mathrm{Z}(G)$ of G equals the kernel of the adjoint representation.

(2) If G is connected, $\mathrm{Z}(G)$ is a closed subgroup and

$$\mathrm{Lie}(\mathrm{Z}(G)) = \mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} \colon \forall Y \in \mathfrak{g}, [X, Y] = 0\}.$$

Exercise 3 (Quotients of Lie groups)**.** Let G be a Lie group and let $K \leq G$ be a closed normal subgroup. Show that G/K can be equipped with a Lie group structure such that the quotient map $\pi\colon G \to G/K$ is a surjective Lie group homomorphism with kernel K.

Exercise 4 (Connectedness from quotients)**.** Let G be a topological group and $H < G$ a closed subgroup. Show that if H and G/H are connected, then so is G.

Exercise 5 (Examples of solvable and nilpotent groups)**.** Compute the derived series and the central series of the Lie groups

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b \neq 0 \right\}, \qquad H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathrm{SL}(2, \mathbb{R})$$

to decide whether they are are solvable and/or nilpotent. Find all weights for the inclusion-representations $\rho_G\colon G \to \mathrm{GL}(2, \mathbb{C})$, $\rho_H\colon H \to \mathrm{GL}(3, \mathbb{C})$ and $\rho_{\mathrm{SL}(2,\mathbb{R})}\colon \mathrm{SL}(2, \mathbb{R}) \to \mathrm{GL}(2, \mathbb{C})$.