Exercise Sheet 7

Exercise 1 (Killing form of $\mathfrak{sl}(2,\mathbb{R})$)**.** Choose a basis of $\mathfrak{sl}(2,\mathbb{R})$ to compute the Killing form $K_{\mathfrak{sl}(2,\mathbb{R})}(X,Y) = 4\mathrm{tr}(XY)$.

Exercise 2 (The Killing form is invariant under the adjoint action)**.** Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Prove that for all $X, Y \in \mathfrak{g}$ and $g \in G$

$$K_{\mathfrak{g}}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = K_{\mathfrak{g}}(X,Y).$$

<u>Hint:</u> Prove that $\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\,\mathrm{ad}(X)\,\mathrm{Ad}(g)^{-1}$.

Exercise 3 (Solvable Lie group without injective finite-dimensional representation)**.** Consider the three-dimensional Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Note that the center of H is

$$\mathrm{Z}(H) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$

Let $D < \mathrm{Z}(H)$ be the following discrete subgroup

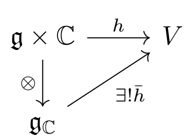
$$D := \mathrm{SL}_3(\mathbb{Z}) \cap \mathrm{Z}(H) = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Check that $G := H/D$ is a connected, solvable Lie group and show that G does not admit a smooth, injective homomorphism into $\mathrm{GL}(V)$ for any finite-dimensional $\mathbb{C}$-vector space V.

<u>Hint:</u> Observe that $\mathrm{Z}(H)/D \cong S^1$ and consider its image under a potential representation and show that its image can be conjugated into any small neighborhood of $\mathrm{Id} \in \mathrm{GL}(n,\mathbb{C})$. Then use the no-small-subgroups property.

Exercise 4 (Complex Lie algebras)**.** Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{g}$ as a vector space.

(1) Show that the bracket $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ extends uniquely to a $\mathbb{C}$-bilinear map $[\cdot,\cdot]_{\mathbb{C}} \colon \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ turning $\mathfrak{g}_{\mathbb{C}}$ into a complex Lie algebra.



(2) Show that the canonical injection $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$, $X \mapsto X \otimes 1$ is a homomorphism of real Lie algebras and, if we identify $\mathfrak{g}$ with its image in $\mathfrak{g}_{\mathbb{C}}$, we have that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}.$$

Express the bracket of $\mathfrak{g}_{\mathbb{C}}$ in this decomposition.

(3) Show that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_{\mathbb{C}}$ is solvable,

(4) Show that $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}_{\mathbb{C}}$ is nilpotent.