Some extra exercises

Here we publish some extra exercises that are not part of this Lie groups course, but are still related to the topics discussed. Exercise 1 gives an example of a topological group whose topology is very different from the Euclidean and the compact-open topology-examples we have seen. Exercise 2 is related to functional analysis. Exercise 3 states that topological groups can be embedded in unimodular topological groups.

Exercise 1 (p-adic Integers $\mathbb{Z}_p$)**.** Let $p \in \mathbb{N}$ be a prime number. Recall that the p-adic integers $\mathbb{Z}_p$ can be seen as the subspace

$$\left\{ (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} : a_{n+1} \equiv a_n \pmod{p^n} \right\}$$

of the infinite product $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}_p$ carrying the induced topology. Note that each $\mathbb{Z}/p^n\mathbb{Z}$ carries the discrete topology and $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is endowed with the resulting product topology.

a) Show that the image of $\mathbb{Z}$ via the embedding

$$\begin{aligned} \iota : \mathbb{Z} &\to \mathbb{Z}_p, \\ x &\mapsto (x \pmod{p^n})_{n\in\mathbb{N}} \end{aligned}$$

is dense. In particular, $\mathbb{Z}_p$ is a compactification of $\mathbb{Z}$.

b) Show that the 2-adic integers $\mathbb{Z}_2$ are homeomorphic to the "middle thirds" cantor set

$$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1].$$

Exercise 2 (Unitary Operators)**.** Let $\mathcal{H}$ be a Hilbert space and $U(\mathcal{H})$ its group of unitary operators. Show that the weak operator topology coincides with the strong operator topology on $U(\mathcal{H})$.

Hint: Recall that a sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *weak operator topology* if

$$\lambda(T_n x) \to \lambda(Tx) \quad (n \to \infty)$$

for every linear functional $\lambda \in \mathcal{H}^*$ and every $x \in \mathcal{H}$.

A sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *strong operator topology* if

$$T_n x \to Tx \quad (n \to \infty)$$

for every $x \in \mathcal{H}$.

Exercise 3. For every locally compact Hausdorff group G, there is a unimodular locally compact Hausdorff group G' and a closed subgroup $H' < G'$ such that $G \cong H'$ are isomorphic.

This exercise is from the book [Nac76], page 100.

Exercise 4. Show that the sphere S^2 does not admit the structure of a Lie group.

Hint: You may use the *hairy ball theorem* without proof: S^2 does not admit a nowhere vanishing vector field.

The following exercise treats a version of the Campbell-Hausdorff-Baker formula.

Exercise 5 (Multiplication and exp)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that for all $X, Y \in \mathfrak{g}$ and small enough $t \in \mathbb{R}$

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + O(t^2))$$

where $O(t^2)$ is a differentiable $\mathfrak{g}$-valued function such that $\frac{O(t^2)}{t^2}$ is bounded as $t \to 0$.

Exercise 6. This exercise illustrates the difference between the Riemannian exponential exp and the Lie group exponential Exp. Let M be a Riemannian manifold and $p \in M$. Then for every $v \in T_pM$ there is a unique geodesic $\gamma_v \colon (-\varepsilon, \varepsilon) \to M$ with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Here, a *geodesic* is a path that locally minimizes distance and is parametrized with constant speed. The *Riemannian exponential* is the map $\exp \colon T_pM \to M$ defined by $\exp(v) = \gamma_v(1)$.

We consider the smooth manifold $M = \mathbb{R}_{>0}$ and the tangent space T_pM at $p = 1 \in \mathbb{R}_{>0} = M$. Consider the Riemannian structures g_1, g_2 on with $g_1(p), g_2(p) \colon T_pM \to \mathbb{R}$, given by $g_1(p)(v,w) = v \cdot w$ and $g_2(p)(v,w) = 1/p^2 \cdot v \cdot w$. Show that they correspond to the metrics $d_1(x,y) = |x-y|$ and $d_2(x,y) = |\log(x/y)|$ on M. Calculate the Riemannian exponential map for the two structures and notice that they don't coincide. Which of these two Riemannian exponentials coincides with the Lie group exponential of the matrix Lie group $(\mathbb{R}_{>0}, \cdot)$?

The following exercise is hard and requires some knowledge of Riemannian geometry and the adjoint representation. A proof can be found in [AB15, Thm. 2.27]. Show that whenever a Lie group is equipped with a Riemanninan structure that is left and right invariant under the group action, then the Riemannian exponential coincides with the Lie group exponential.

Hint: Along the way you might want to prove: Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group with bi-invariant metric. If $X, Y, Z \in \mathfrak{g}$, then $\langle [X,Y], Z\rangle = -\langle [X,Z], Y\rangle$. If ∇ is the Levi-Civita connection, then for $X, Y \in \mathfrak{g}$ holds $\nabla_X Y = \frac{1}{2}[X,Y]$.

Exercise 7. A topological group has *small subgroups* if every neighborhood of the identity contains a non-trivial subgroup.

Show that Lie groups have no small subgroup.

In fact the following is an answer to Hilbert's 5th question:

Let G be a connected locally compact topological group. Then G is a Lie group if and only if G has no small subgroups.

Exercise 8. (No exotic Lie groups)

A second countable locally Euclidean topological group has at most one smooth structure making it into a Lie group.

Hint: Use Exercise 3(b) on sheet 5.

References

[AB15] Marcos M. Alexandrino, Renato G. Bettiol. *Lie groups and geometric aspects of isometric actions.* Springer, Cham, 2015.

[Nac76] Leopoldo Nachbin. *The Haar integral.* Robert E. Krieger Publishing Co., Huntington, NY, 1976. Translated from the Portuguese by Lulu Bechtolsheim, Reprint of the 1965 edition.