Solutions to Exercise Sheet 1

Exercise 1 (Product of topological groups)**.** Let A be a set and for every $\alpha \in A$, G_α a topological group. Show that

$$G := \prod_{\alpha \in A} G_\alpha$$

with the product topology[1] is a topological group.

Solution. We first consider the multiplication $m\colon G \times G \to G, ((g_\alpha)_\alpha, (h_\alpha)_\alpha) \mapsto (g_\alpha h_\alpha)_\alpha$. Let $O = \prod U_\alpha$ be an open set of the basis of the product topology; this means U_α open and $U_\alpha = G_\alpha$ for all but finitely many $\alpha \in A$. Now

$$\begin{aligned} m^{-1}(O) &= \{(g,h) \in G \times G \colon gh \in O\} = \{((g_\alpha)_\alpha, (h_\alpha)_\alpha) \in G \times G \colon g_\alpha h_\alpha \in U_\alpha\} \\ &\cong \{(g_\alpha, h_\alpha)_\alpha \in \prod_\alpha (G_\alpha \times G_\alpha) \colon g_\alpha h_\alpha \in U_\alpha\} = \prod_\alpha \{(g_\alpha, h_\alpha) \in G_\alpha \times G_\alpha \colon g_\alpha h_\alpha \in U_\alpha\} \\ &= \prod_\alpha m_\alpha^{-1}(U_\alpha). \end{aligned}$$

where we identified $G \times G \cong \prod(G_\alpha \times G_\alpha)$. Since G_α are topological groups $m_\alpha^{-1}(U_\alpha) \subseteq G_\alpha \times G_\alpha$ open. For all but finitely many $\alpha \in A$, $U_\alpha = G_\alpha$, so $m_\alpha^{-1}(U_\alpha) = G_\alpha \times G_\alpha$, so $m^{-1}(O)$ is open in $\prod(G_\alpha \times G_\alpha) \cong G \times G$.

The inversion $i\colon G \times G, (g_\alpha)_\alpha \mapsto (g_\alpha^{-1})_\alpha$ satisfies $i^{-1}(O) = \prod U_\alpha^{-1}$. Since $i_\alpha\colon G_\alpha \to G_\alpha$ is continuous, $U_\alpha^{-1} = i_\alpha^{-1}(U_\alpha)$ is open and equal to G_α for all but finitely many $\alpha \in A$. Hence $i^{-1}(O)$ is open.

A general open set of G is a union $\cup O_i$ of open sets of the basis. We note that $m^{-1}(\cup O_i) = \cup m^{-1}(O_i)$ and $i^{-1}(\cup O_i) = \cup i^{-1}(O_i)$, so the preimages of any open set are open. This means that m and i are continuous, and G is a topological group.

Exercise 2 ($\mathrm{O}(p,q)$)**.** We consider the orthogonal group $\mathrm{O}(p,q)$ of signature $p, q \geq 1$.

a) Show that the connected component of the group $O(1,1)$ containing the identity is homeomorphic to $\mathbb{R}$.

b) Show that for all $p, q \geq 1$, $\mathrm{O}(p,q)$ has a subgroup isomorphic to $\mathbb{R}$.

Solution. a) We recall that

$$\mathrm{O}(1,1) = \left\{ g \in \mathrm{GL}(2,\mathbb{R}) \colon {}^{\mathrm{t}}g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

[1] A basis of the product topology is given by the sets $\prod U_\alpha$, where U_α open and $U_\alpha = G_\alpha$ for all but finitely many $\alpha \in A$.

Now if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{O}(1,1),$$

we obtain the conditions $a^2 - c^2 = 1, d^2 - b^2 = 1$ and $ab = cd$. Rephrasing $a = cd/b$ and $b = cd/a$ we obtain

$$\begin{aligned} & \left(\frac{cd}{b}\right)^2 - c^2 = 1 \quad \text{and} \quad d^2 - \left(\frac{cd}{a}\right)^2 = 1 \\ \iff & \quad c^2d^2 - c^2b^2 = b^2 \quad \text{and} \quad a^2d^2 - c^2d^2 = a^2 \\ \iff & \quad 1 = d^2 - b^2 = b^2/c^2 \quad \text{and} \quad 1 = a^2 - c^2 = a^2/d^2 \\ \iff & \quad b^2 = c^2 \quad \text{and} \quad a^2 = d^2, \end{aligned}$$

so $a = \pm d$ and $b = \pm c$. By $ab = cd$, both signs have to be the same. We obtain that

$$\mathrm{O}(1,1) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})\colon a^2 - b^2 = 1 \right\} \cup \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})\colon a^2 - b^2 = 1 \right\}.$$

Every a, b with $a^2 - b^2 = 1$ can be written as $a = \pm\cosh(\varphi)$ and $b = \sinh(\varphi)$ for some unique $\varphi \in \mathbb{R}$. Thus

$$\begin{aligned} \mathrm{O}(1,1) = & \left\{ \begin{pmatrix} \cosh(\varphi) & \sinh(\varphi) \\ \sinh(\varphi) & \cosh(\varphi) \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} -\cosh(\varphi) & \sinh(\varphi) \\ \sinh(\varphi) & -\cosh(\varphi) \end{pmatrix} \right\} \cup \\ & \left\{ \begin{pmatrix} \cosh(\varphi) & \sinh(\varphi) \\ -\sinh(\varphi) & -\cosh(\varphi) \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} -\cosh(\varphi) & \sinh(\varphi) \\ -\sinh(\varphi) & \cosh(\varphi) \end{pmatrix} \right\}. \end{aligned}$$

The description of $\mathrm{O}(1,1)$ above shows that there are four parts of $\mathrm{O}(1,1)$, all of which are pathconnected, (parametrize the paths using φ). We claim that the four parts are distinct connected components:

Note that the determinant on the first two parts is $a^2 - b^2 = 1$ and the determinant on the last two parts is $-a^2 + b^2 = -1$. Since the determinant is a continuous map $\mathrm{O}(1,1) \to \mathbb{R}$ this shows that those parts are in different components. To distinguish more, we consider the continuous map $g = (g_{ij})_{ij} \mapsto g_1 1$. On the first and third part this function is at least 1, while on the second and third component, this function is at most -1. These two observations imply that all four parts are contained in different connected components. Since they are path-connected, they are exactly the four connected components of $O(1,1)$.

We note that the first component contains Id when $\varphi = 0$, so the connected component of the identity is

$$\mathrm{O}(1,1)^\circ = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})\colon a^2 - b^2 = 1 \right\} = \left\{ \begin{pmatrix} \cosh(\varphi) & \sinh(\varphi) \\ \sinh(\varphi) & \cosh(\varphi) \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})\colon \varphi \in \mathbb{R} \right\}$$

and the last description shows that it is homeomorphic to $\mathbb{R}$.

b) Equipped with the ideas from part a), we consider the subgroup

$$G = \{g(\varphi)\colon \varphi \in \mathbb{R}\} \qquad \text{for} \qquad g(\varphi) = \left(\begin{array}{ccc|ccc} \cosh(\varphi) & 0 & \cdots & \sinh(\varphi) & 0 & \cdots \\ 0 & 1 & \ddots & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots \\ \hline \sinh(\varphi) & 0 & \cdots & \cosh(\varphi) & 0 & \cdots \\ 0 & 0 & \ddots & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots \end{array}\right)$$

Explicit calculations show that $g(\varphi) \in \mathrm{O}(p, q)$.

Exercise 3 (Compact-Open Topology)**.** Let X, Y, Z be a topological space, and denote by $C(Y, X) :=$ $\{f\colon Y \to X \text{ continuous}\}$ the set of continuous maps from Y to X. The set $C(Y, X)$ can be endowed with the *compact-open topology*, that is generated by the subbasic sets

$$S(K, U) := \{f \in C(Y, X) \,|\, f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.

Prove the following useful facts about the compact-open topology.

If Y is locally compact[2] , then:

a) The evaluation map $e\colon C(Y, X) \times Y \to X, e(f, y) := f(y)$, is continuous.

b) A map $f\colon Y \times Z \to X$ is continuous if and only if the map

$$\hat{f}\colon Z \to C(Y, X), \quad \hat{f}(z)(y) = f(y, z),$$

is continuous.

Solution. a) For $(f, y) \in C(Y, X) \times Y$ let $U \subset X$ be an open neighborhood of $f(y)$. Since Y is locally compact, continuity of f implies there is a compact neighborhood $K \subset Y$ of y such that $f(K) \subset U$. Then $S(K, U) \times K$ is a neighborhood of (f, y) in $C(Y, X) \times Y$ taken to U by e, so e is continuous at (f, y).

b) Suppose $f\colon Y \times Z \to X$ is continuous. To show continuity of $\hat{f}$ it suffices to show that for a subbasic set $S(K, U) \subset C(Y, X)$, the set $\hat{f}^{-1}(S(K, U)) = \{z \in Z \,|\, f(K, z) \subset U\}$ is open in Z. Let $z \in \hat{f}^{-1}(S(K, U))$. Since $f^{-1}(U)$ is an open neighborhood of the compact set $K \times \{z\}$, there exist open sets $V \subset Y$ and $W \subset Z$ whose product $V \times W$ satisfies $K \times \{z\} \subset V \times W \subset f^{-1}(U)$. Indeed, $f^{-1}(U) = \cup_{i \in I} V_i \times W_i$ and we can choose a finite family $I' \subset I$ with $K \times \{z\} \subset \cup_{i \in I'} V_i \times W_i$. Then set $W := \cap_{z \in W_i} W_i$ and $V := \cup_{z \in W_i} V_i$.

[2]A subset $C \subseteq Y$ that contains an open subset $U \subseteq Y$ with $y \in U \subseteq C \subseteq Y$ is called a *neighborhood of* $y \in Y$. Then Y is called *locally compact* if for every $y \in Y$ there is a set $\mathcal{D}$ of compact neighborhoods of y such that every neighborhood of y contains an element of $\mathcal{D}$ as a subset.

So W is a neighborhood of z in $\hat{f}^{-1}(S(K,U))$. (The hypothesis that Y is locally compact is not needed here.)

For the converse of b) note that f is the composition $Y \times Z \to Y \times C(Y,X) \to X$ of $\mathrm{Id} \times \hat{f}$ and the evaluation map, so part a) gives the result.

Exercise 4 (General Linear Group $\mathrm{GL}(n,\mathbb{R})$)**.** The general linear group

$$\mathrm{GL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\} \subseteq \mathbb{R}^{n\times n}$$

is naturally endowed with the subspace topology of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$. However, it can also be seen as a subset of the space of homeomorphisms of $\mathbb{R}^n$ via the injection

$$\begin{aligned} j\colon\ \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{Homeo}(\mathbb{R}^n), \\ A &\mapsto (x \mapsto Ax). \end{aligned}$$

a) Show that $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ is a closed subset, where $\mathrm{Homeo}(\mathbb{R}^n) \subset C(\mathbb{R}^n,\mathbb{R}^n)$ is endowed with the compact-open topology.

Solution. Note that

$$j(\mathrm{GL}(n,\mathbb{R})) = \{f \in \mathrm{Homeo}(\mathbb{R}^n) : f(\lambda x + y) = \lambda f(x) + f(y) \text{ for all } \lambda \in \mathbb{R}, x, y \in \mathbb{R}^n\}.$$

Since evaluation is continuous also the maps

$$\begin{aligned} F_{\lambda,x,y} : \mathrm{Homeo}(\mathbb{R}^n) &\to \mathbb{R}^n \\ f &\mapsto f(\lambda x + y) - \lambda f(x) - f(y) \end{aligned}$$

are continuous for all $\lambda \in \mathbb{R}, x, y \in \mathbb{R}^n$.

Thus,

$$j(\mathrm{GL}(n,\mathbb{R})) = \bigcap_{\lambda\in\mathbb{R}, x,y\in X} F^{-1}_{\lambda,x,y}(0) \subset \mathrm{Homeo}(\mathbb{R}^n)$$

is closed as the intersection of closed sets.

b) If we identify $\mathrm{GL}(n,\mathbb{R})$ with its image $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ we can endow it with the induced subspace topology. Show that this topology coincides with the usual topology coming from the inclusion $\mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$. <u>Hint:</u> Exercise 3 can be useful here.

Solution. Consider the inclusions

$$\begin{aligned} i : \mathrm{GL}(n,\mathbb{R}) &\to \mathbb{R}^{n\times n}, \\ A &\mapsto \begin{pmatrix} | & & | \\ A\mathbf{e}_1 & \cdots & A\mathbf{e}_n \\ | & & | \end{pmatrix}, \end{aligned}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denotes the standard basis of $\mathbb{R}^{n\times n}$.

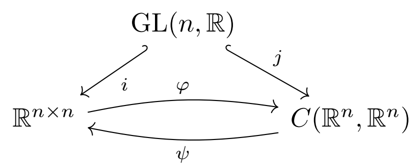
Further, consider the maps

$$\begin{aligned} \varphi : \mathbb{R}^{n\times n} &\to C(\mathbb{R}^n,\mathbb{R}^n), \\ \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} &\mapsto (\mathbf{x} \mapsto x_1 \cdot \mathbf{v}_1 + \cdots + x_n \cdot \mathbf{v}_n), \end{aligned}$$

and

$$\psi : C(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^{n\times n},$$
$$f \mapsto \begin{pmatrix} | & & | \\ f(\mathbf{e}_1) & \cdots & f(\mathbf{e}_n) \\ | & & | \end{pmatrix}.$$

It is easy to verify that these form the following commutative diagram.



Since both topologies under consideration on $\mathrm{GL}(n, \mathbb{R})$ come from pulling back the topologies of $\mathbb{R}^{n\times n}$ resp. $C(\mathbb{R}^n, \mathbb{R}^n)$ via i resp. j they will coincide if we can show that the maps φ and ψ are continuous[3].

The map ψ is continuous because it is the product of the evaluation maps

$$\mathrm{ev}_{\mathbf{e}_i} : C(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n, \mathrm{ev}_{\mathbf{e}_i}(f) = f(\mathbf{e}_i)$$

$(i = 1, \ldots, n)$.

Further, observe that the map

$$\mathrm{ev} \circ (\varphi \times \mathrm{Id}) : \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^n, (A, x) \mapsto Ax$$

is continuous. This implies that φ is continuous.

Exercise 5 (Isometry Group $\mathrm{Iso}(X)$)**.** Let (X, d) be a *compact* metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in X\}.$$

Show that $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is compact with respect to the compact-open topology.

Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem, see Appendix A.2 in Prof. Alessandra Iozzi's book.

Solution. The compact-open topology on $\mathrm{Homeo}(X)$ coincides with the topology induced by the metric of uniform convergence

$$d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

[3] Let τ_i, τ_j denote the topologies, so that τ_i is the smallest topology on $\mathrm{GL}(n, \mathbb{R})$ such that i is continuous and τ_j is the smallest such that j is continuous. If φ is continuous, then

$$j = \varphi \circ i : (\mathrm{GL}(n, \mathbb{R}), \tau_i) \to C(\mathbb{R}^n, \mathbb{R}^n)$$

is continuous, thus $\tau_j \subset \tau_i$. Analogously, if ψ is continuous, then $\tau_i \subset \tau_j$ and so the two topologies coincide.

Note that by Arzelà–Ascoli (Theorem A.1 in the lecture notes) a family $\mathcal{F} \subseteq C(X,X)$ of continuous maps is compact if and only if $\mathcal{F}$ is equicontinuous, and $\mathcal{F}$ is closed.

Equicontinuity of $\mathcal{F} := \mathrm{Iso}(X)$ is clear, because we are dealing with isometries. We check that $\mathrm{Iso}(X)$ is closed.

Let $f \in C(X,X)$ and let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Iso}(X)$ be a sequence converging to it. Let $x, y \in X$ then

$$\begin{aligned} 0 &\leq |d(f(x), f(y)) - d(x,y)| \\ &= |d(f(x), f(y)) - d(f_n(x), f_n(y))| \\ &\leq |d(f(x), f(y)) - d(f_n(x), f(y))| + |d(f_n(x), f(y)) - d(f_n(x), f_n(y))| \\ &\leq d(f(x), f_n(x)) + d(f(y), f_n(y)) \to 0 \quad (n \to \infty). \end{aligned}$$

Hence, f is an isometry as wished for. Because f was arbitrary this shows that $\mathrm{Iso}(X) \subseteq C(X,X)$ is closed.

Exercise 6 (Homeo($\mathbb{S}^1$) is not locally compact.)**.** Let $\mathbb{S}^1 \subseteq \mathbb{C} \setminus \{0\}$ denote the circle. Show that Homeo($\mathbb{S}^1$) with the compact-open topology is not locally compact.

Solution. We will prove a more general fact, namely that Homeo(M) is not locally compact for any compact manifold M of positive dimension. Note that we can think of M as a compact metric space (M,d) by Urysohn's metrization theorem. In the case when M is a smooth manifold this is even easier to see by endowing it with a Riemannian metric. This puts us now in the favorable position of being able to identify the compact-open topology on Homeo(X) with the topology of uniform convergence.

We denote by

$$d_\infty(f,g) := \sup\{d(f(x), g(x)) : x \in M\}$$

the metric of uniform convergence on Homeo(M). Further denote by $B_f^\infty(r)$ the ball of radius $r > 0$ about a homeomorphism $f \in \mathrm{Homeo}(M)$. In order to show that Homeo(M) is not locally compact we will construct in every $\varepsilon > 0$ ball about the identity $B_{\mathrm{Id}}^\infty(\varepsilon)$ a sequence of homeomorphisms $(f_k)_{k\in\mathbb{N}}$ with no convergent subsequence.

Let $\varepsilon > 0$ and denote $B = B_{\mathrm{Id}}^\infty(\varepsilon)$. Further, let $x_0 \in M$ and choose a coordinate chart $\varphi : U \subset B_{\varepsilon/2}(x_0) \to \mathbb{R}^n$ centered at x_0 (i.e. $\varphi(x_0) = 0$) contained in the $\varepsilon/2$-ball $B_{\varepsilon/2}(x_0)$ about x_0 in M. Consider the homeomorphisms

$$\psi_k : \overline{B_1}(0) \to \overline{B_1}(0), \quad x \mapsto \|x\|^k x$$

on the closed unit ball $\overline{B_1}(0)$ in $\mathbb{R}^n$ which fix $0 \in \mathbb{R}^n$ and the boundary n-sphere pointwise. Note that the sequence $(\psi_k)_{k\in\mathbb{N}}$ converges pointwise to

$$\psi_\infty = \begin{cases} x, & \text{if } x \in \partial B_1(0), \\ 0, & \text{if } x \in B_1(0). \end{cases}$$

Now, define

$$f_k(x) := \begin{cases} x, & \text{if } x \notin \varphi^{-1}(B_1(0)), \\ \varphi^{-1}(\psi_k(\varphi(x))), & \text{if } x \in \varphi^{-1}(B_1(0)). \end{cases}$$

It is easy to see that the maps $f_k : M \to M$ are indeed homeomorphisms: $f_k|_{\varphi^{-1}(\overline{B_1}(0))^c} = \mathrm{Id} : \varphi^{-1}(\overline{B_1}(0))^c \to \varphi^{-1}(\overline{B_1}(0))^c$ is a homeomorphism, $\varphi^{-1} \circ \psi_k \circ \varphi : \varphi^{-1}(\overline{B_1}(0)) \to \varphi^{-1}(\overline{B_1}(0))$ is a homeomorphism and both coincide on $\varphi^{-1}(\partial B_1(0))$.

Further, the homeomorphisms f_k map the $\varepsilon/2$-ball $B_{\varepsilon/2}(x_0)$ to itself and fix x_0. Therefore,

$$d(f_k(x), x) \leq d(f_k(x), \underbrace{f_k(x_0)}_{=x_0}) + d(x_0, x) < \varepsilon,$$

for every $x \in B_{\varepsilon/2}(x_0)$, and clearly $f_k(x) = x$ for every $x \notin B_{\varepsilon/2}(x_0)$. Hence, the sequence $(f_k)_{k\in\mathbb{N}}$ is in $B^\infty_\varepsilon(\mathrm{Id})$.

However, the sequence $(f_k)_{k\in\mathbb{N}}$ converges pointwise to

$$f_\infty(x) = \begin{cases} x, & \text{if } x \notin \varphi^{-1}(B_1(0)), \\ x_0, & \text{if } x \in \varphi^{-1}(B_1(0)), \end{cases}$$

If there were a subsequence $(f_{k_l})_{l\in\mathbb{N}}$ converging to some $f \in \mathrm{Homeo}(M)$ uniformly then this sequence would also converge pointwise to f, i.e. f needs to coincide with f_∞. But f_∞ is not even continuous which contradicts our assumption of $f \in \mathrm{Homeo}(M)$. Therefore $(f_k)_{k\in\mathbb{N}} \subset B^\infty_\varepsilon(\mathrm{Id})$ has no uniformly convergent subsequences.

Exercise 7 (Coverings of topological groups)**.** Let H be a topological group, G a topological space and $p\colon G \to H$ a covering[4]. Assume that both H and G are path-connected and locally path-connected. Show that for every $\tilde{e} \in p^{-1}(e_H)$ there is a unique topological group structure on G such that $\tilde{e}$ is the neutral element and p is a group homomorphism.

Hint: You may use the *lifting criterion*: If $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering and $f\colon (Y, y_0) \to (X, x_0)$ is a continuous map, where Y is path-connected and locally path-connected, then there is a unique continuous lift $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f, i.e. $p \circ \tilde{f} = f$, if and only if $f_\star(\pi_1(Y, y_0)) \subseteq p_\star(\pi_1(\tilde{X}, \tilde{x}_0))$.

Solution. We will lift the multiplication and inversion maps to G and show that they define a group structure on G.

Let $m : H \times H \to H$ and $i : H \to H$ denote the multiplication and inversion maps of H, respectively, and let e_G be an arbitrary element of the fiber $p^{-1}(e_H) \subseteq G$.

$$\begin{array}{ccc} G \times G & \xrightarrow{\tilde{m}} & G \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ H \times H & \xrightarrow{m} & H \end{array}$$

[4]A *covering* $p\colon G \to H$ is a continuous map such that for every $h \in H$ there is an open neighborhood $U_h \subseteq H$ and a discrete space D_h such that $p^{-1}(x) = \amalg_{d\in D_h} V_d$ and for every $d \in D_h$, $p|_{V_d}\colon V_d \to U_h$ is a homeomorphism.

We note that $m \circ (p \times p) : G \times G \to H \times H$ is a continuous map and if we want to lift it to $\tilde{m}$, it needs to satisfy $(m \circ (p \times p))_\star(\pi_1(G \times G, (e_G, e_G))) \subseteq p_\star(\pi_1(G, e_G))$.

Lemma: $m_\star([\gamma], [\alpha]) = [\gamma] \circ [\alpha]$ for all loops γ, α in H based at e_H, where $\circ$ is the group operation in in $\pi_1(H, e_H)$.

Proof: If $1 \colon t \mapsto e_H$ denotes the trivial loop in H, we have by definition that $m_\star([\gamma], [1])$ is the equivalence class of the loop $[0,1] \to H, t \mapsto \gamma(t) \cdot e_H = \gamma(t)$, so $m_\star([\gamma], [1]) = [\gamma]$ and similarly $m_\star([1], [\alpha]) = [\alpha]$. Now since $m_\star$ is a group homomorphism

$$m_\star([\gamma], [\alpha]) = m_\star(([\gamma], [1])([1], [\alpha])) = m_\star([\gamma], [1]) \circ m_\star([1], [\alpha]) = [\gamma] \circ [\alpha].$$

Now given any $[\gamma], [\alpha] \in \pi_1(G)$, we have $(m \circ (p \times p))_\star([\gamma], [\alpha]) = m_\star(p_\star([\gamma]), p_\star([\alpha])) = p_\star[\gamma] \circ p_\star[\alpha]$ and this is contained in the image of $p_\star$ since

$$p_\star([\gamma \circ \alpha]) = p_\star([\gamma] \circ [\alpha]).$$

The map $m \circ (p \times p) : G \times G \to H$ thus has a unique continuous lift $\tilde{m} : G \times G \to G$ satisfying $\tilde{m}(e_G, e_G) = e_G$ and $p \circ \tilde{m} = m \circ (p \times p)$.

By similar reasoning, $i \circ p : G \to H$ has a smooth lift $\tilde{i} : G \to G$ satisfying $\tilde{i}(e_G) = e_G$ and $p \circ \tilde{i} = i \circ p$:

$$\begin{array}{ccc} G & \xrightarrow{\tilde{i}} & G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ H & \xrightarrow{i} & H \end{array}$$

We define multiplication and inversion in G by $xy = \tilde{m}(x, y)$ and $x^{-1} = \tilde{i}(x)$. By the above commutative diagrams we obtain

$$p(xy) = p(x)p(y), \qquad p(x^{-1}) = p(x)^{-1}.$$

It remains to show that G is a group with these operations, for then it is a topological group because $\tilde{m}$ and $\tilde{i}$ are continuous and the above relations imply that p is a homomorphism.

First we show that e_G is an identity for multiplication in G. Consider the map $f : G \to G$ defined by $f(x) = e_G x$. Then

$$p(f(x)) = p(e_G)p(x) = e_H p(x) = p(x),$$

so f is a lift of $p : G \to H$. The identity map Id_G is another lift of p, and it agrees with f at a point because $f(e_G) = \tilde{m}(e_G, e_G) = e_G$, so the unique lifting property of covering maps implies that $f = \mathrm{Id}_G$, or equivalently, $e_G x = x$ for all $x \in G$. The same argument shows that $x e_G = x$.

Next, to show that multiplication in G is associative, consider the two maps $\alpha_L, \alpha_R : G \times G \times G \to G$ defined by

$$\alpha_L(x, y, z) = (xy)z, \quad \alpha_R(x, y, z) = x(yz).$$

Then

$$p(\alpha_L(x, y, z)) = (p(x)p(y))p(z) = p(x)(p(y)p(z)) = p(\alpha_R(x, y, z)),$$

so α_L and α_R are both lifts of the same map $\alpha(x, y, z) = p(x)p(y)p(z)$. Because α_L and α_R agree at (e_G, e_G, e_G), they are equal. A similar argument shows that $x^{-1}x = xx^{-1} = e_G$, so G is a group.

The uniqueness follows from the uniqueness of the lifting property.