Solutions to Exercise Sheet 3

Exercise 1. For each of the following locally compact Hausdorff groups, give an example of a lattice or prove that it does not admit a lattice.

a) The free group on 2 generators with the discrete topology.

Solution. A lattice $\Gamma < G$ in a locally compact Hausdorff group G is a discrete subgroup such that G/Γ admits a finite G-invariant regular Borel measure. For discrete groups, such as the free group on two generators $F_2 = \langle a, b\rangle$, we can take $\Gamma = F_2$, with G/Γ consisting of a single point. A single point clearly admits a finite regular Borel measure, namely the one that associates to $\emptyset$ the value 0 and to G/Γ some positive value, such as 1.

b) $G = \mathrm{SO}(n, \mathbb{R})$.

Solution. $G = \mathrm{SO}(n, \mathbb{R})$ is compact, which can for example be seen by the fact that it is bounded and closed in the Euclidean topology. We can consider the discrete subgroup $\Gamma = \{\mathrm{Id}\}$ to obtain $G/\Gamma = \mathrm{SO}(n, \mathbb{R})$. $\mathrm{SO}(n, \mathbb{R})$ admits a Haar-measure (since it is compact, the Haar measure is both a left and a right Haar measure), which is a $\mathrm{SO}(n, \mathbb{R})$-invariant regular Borel measure on itself. The measure is finite, since by definition of the Haar measure, compactly supported functions are assigned finite values by the positive linear functional $\Lambda\colon C_c(G) \to \mathbb{C}$ associated to the Haar measure. Thus $\{\mathrm{Id}\}$ is a lattice.

c) $G = (\mathbb{R}_{>0}, \cdot)$.

Solution. Let e be Euler's number. Let $\Gamma = \{e^z \in \mathbb{R}_{>0} \colon z \in \mathbb{R}\}$, then Γ is a subgroup, since $e^{z_1}e^{z_2} = e^{z_1+z_2}$ isomorphic to $\mathbb{Z}$. Γ is discrete, as every singleton $\{e^z\}$ for some $z \in \mathbb{Z}$ is open in the subset topology (take an open ball with radius $e^z - e^{z-1}$).

The group $\mathbb{R}_{>0}$ has the Haar measure $\frac{1}{x}dx$ (and is unimodular, since abelian). By the Weyl formula (Theorem 2.68), there is a G-invariant positive regular Borel measure on the homogeneous space G/Γ if and only if the modular functions of G and Γ satisfy $\Delta_G|_\Gamma = \Delta_\Gamma$. Since both G and Γ are abelian, Δ_G and Δ_Γ are both constant 1, so the Weyl formula can be applied, which also gives us

$$\int_{G/\Gamma}\int_H f(gh)\,\mathrm{d}\mu_\Gamma(h)\,\mathrm{d}\mu_{G/\Gamma}(g\Gamma) = \int_G f(g)\,\mathrm{d}\mu_G(g)$$

for any compactly supported function $f \in C_c(G)$. We may pick $f = \chi_{[1,e)}$ the characteristic function of the subset $[1, e) \subseteq G = \mathbb{R}_{>0}$. The subset $[1, e)$ is called a *fundamental domain* for the action of Γ on G, since for every $g \in G$, there is exactly one $h_0 \in \Gamma$ such that $gh_0 \in [1, e)$. We thus have for every $g \in G$,

$$\int_H \chi_{[1,e)}(gh)\,\mathrm{d}\mu_H(h) = \int_H \chi_{[g^{-1},g^{-1}e)}(h)\,\mathrm{d}\mu_H(h) = 1 \cdot \mu_H(\{h_0\}),$$

which is a finite constant, since μ_H is H-invariant. Now taking $f = \frac{1}{\mu_\Gamma(\{h_0\})}\chi_{[1,e)}$, we have

$$\mu_{G/\Gamma}(G/\Gamma) = \int_{G/\Gamma} 1 \,\mathrm{d}\mu_{G/\Gamma}(g\Gamma) = \int_{G/\Gamma} \frac{1}{\mu_\Gamma(\{h_0\})} \cdot \int_\Gamma \chi_{[1,e)}(gh)\,\mathrm{d}\mu_\Gamma(h)\,\mathrm{d}\mu_{G/\Gamma}(h\Gamma)$$
$$= \int_{G\Gamma}\int_\Gamma f(gh)\,\mathrm{d}\mu_\Gamma(h)\,\mathrm{d}\mu_{G/\Gamma}(h\Gamma) = \int_G f(g)\,\mathrm{d}\mu_G(g) < 0$$

since f is compactly supported. This means that the measure on G/Γ is finite and Γ is a lattice.

In fact, using the compatible isomorphisms $\mathbb{R}_{>0} \cong \mathbb{R}$, $\Gamma \cong \mathbb{Z}$, we obtain $G/\Gamma \overset{\text{homeo}}{=} \mathbb{R}/\mathbb{Z} = S^1$, which is compact, and therefore has finite measure.

d) $G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$, see exercise 2b) on Sheet 2.

Solution. By Sheet 2, Exercise 2b), we know that G is not unimodular. However, Proposition 2.70 states that if a locally compact Hausdorff group admits a lattice, then it is unimodular. We conclude that G does not admit a lattice.

e) The Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$

see exercise 2a) on Sheet 2.

Solution. We consider the integer points

$$\Gamma := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\} < G,$$

and notice that Γ is a group and discrete. It remains to show that G/Γ admits a finite G-invariant regular Borel measure. Even though G is not abelian, it is unimodular, see Sheet 2, exercise 2a). Since Γ is a discrete group, it is also unimodular. Hence $\Delta_G|_\Gamma = \Delta_\Gamma$, and by the Weyl formula (Theorem 2.68), there is a G-invariant regular Borel measure $\mu_{G/\Gamma}$ on G/Γ. We claim that the subset

$$R := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in [0, 1) \right\} \subseteq G$$

is a fundamental region of the action of Γ on G. Indeed, for any $g \in G$, we can find an $h_g \in \Gamma$ such that $h_g g \in R$, by applying first

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{then} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and finally} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

from the left. We also note that h_g is unique, as for every $h \in \Gamma$ and $r \in R$, if $hr \in R$, then $h = \mathrm{Id}$. Now the measure of the fundamental domain is (up to a positive multiplication) given by the Haar measure of R, as can be calculated using the Weyl formula

$$\int_{G/\Gamma} \int_H f(gh)\, \mathrm{d}\mu_\Gamma(h)\, \mathrm{d}\mu_{G/\Gamma}(g\Gamma) = \int_G f(g)\, \mathrm{d}\mu_G(g)$$

applied to the characteristic function $f = \chi_R$.

Exercise 2 (Regular Subgroups are closed)**.** Let G be a Lie group, $H \leq G$ a subgroup that is also a regular submanifold. Prove that H is a closed subgroup of G.

Solution. Let $x \in \overline{H}$. As G is clearly first countable, we find $(x_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ such that $x = \lim_{n\to\infty} x_n$. Let $V \subseteq W \subseteq \overline{W} \subseteq U$ open neighbourhoods of $1 \in G$ with compact closure and assume that $\psi : U \to (-1,1)^{\dim G}$ is a chart as in the definition of a regular submanifold. Assume furthermore that V is symmetric and $VV \subseteq W$. By assumption, there is $N \geq 1$ such that $x_n \in xV$ for all $n \geq N$, thus $x_N^{-1} x_n \in V^{-1}x^{-1}xV = VV \subseteq W$ for all $n \geq N$, and thus $x_N^{-1}x_n \in H \cap VV \subseteq H \cap \overline{W}$. We note that $H \cap \overline{W}$ is compact by the choice of U. Indeed, $\psi(\overline{W}) \subseteq (-1,1)^{\dim G}$ is compact, and so is $\psi(\overline{W}) \cap \{0\}^{\dim G - \dim H} \times (-1,1)^{\dim H}$. But $x_N^{-1}x_n$ is convergent and has a limit y in $H \cap \overline{W}$; whence $x_N y = x \in H$.

Exercise 3 (The Matrix Lie Group $O(p,q)$)**.** Let $p, q \in \mathbb{N}$ and $n = p+q$. We define the (indefinite) symmetric bilinear form $\langle \cdot,\cdot \rangle_{p,q}$ of signature (p,q) on $\mathbb{R}^n$ to be

$$\langle v, w\rangle_{p,q} := v_1w_1 + \cdots + v_pw_p - v_{p+1}w_{p+1} - \cdots - v_{p+q}w_{p+q}$$

for all $v = (v_1,\ldots,v_n), w = (w_1,\ldots,w_n) \in \mathbb{R}^n$. Show that

$$O(p,q) := \{A \in \mathrm{GL}(n,\mathbb{R}) : \langle Av, Aw\rangle_{p,q} = \langle v,w\rangle_{p,q} \quad \forall v,w \in \mathbb{R}^n\}.$$

is a Lie group using the inverse function theorem/constant rank theorem. What is its dimension?

Solution. We define

$$I_{p,q} := \mathrm{diag}(\underbrace{1,\ldots,1}_{p\text{-times}}, \underbrace{-1,\ldots,-1}_{q\text{-times}})$$

to be the diagonal matrix that has $+1$ in the first p entries along the diagonal and -1 in the last q entries. It is easy to see that

$$O(p,q) := \left\{A \in \mathrm{GL}(n,\mathbb{R}) : A^T I_{p,q} A = I_{p,q}\right\}.$$

Now, define

$$f : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}, A \mapsto A^T I_{p,q} A,$$

such that $O(p,q) = f^{-1}(I_{p,q})$. The map f is smooth as every entry of $f(A)$ is a polynomial in the entries of $A \in \mathrm{GL}(n,\mathbb{R})$.

We proceed by showing that f has constant rank. Let $X \in T_A \mathrm{GL}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n}, A \in \mathrm{GL}(n,\mathbb{R})$. We compute directly

$$\begin{aligned} D_A f(X) &= \frac{d}{dt}\bigg|_{t=0} (A+tX)^T I_{p,q}(A+tX) \\ &= \frac{d}{dt}\bigg|_{t=0} \left(A^T I_{p,q}A + t\cdot X^T I_{p,q}A + t\cdot A^T I_{p,q}X + t^2\cdot X^T I_{p,q}X\right) \\ &= X^T I_{p,q}A + A^T I_{p,q}X = (A^T I_{p,q}X)^T + A^T I_{p,q}X. \end{aligned}$$

We claim that the image consists of all symmetric matrices $\mathrm{Sym}_n(\mathbb{R}) \subset \mathbb{R}^{n\times n}$ and that $D_A f : T_A \mathrm{GL}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n} \to \mathrm{Sym}_n(\mathbb{R})$ is onto. For that consider the projection

$$\begin{aligned} p : \mathbb{R}^{n\times n} &\to \mathrm{Sym}_n(\mathbb{R}), \\ X &\mapsto \tfrac{1}{2}\left(X + X^T\right). \end{aligned}$$

It is easy to check that $p\circ p = p$ and $p|_{\mathrm{Sym}_n(\mathbb{R})} = \mathrm{Id}$, such that p is onto. Since,

$$D_A f(X) = 2\cdot p(A^T I_{p,q}X)$$

and A is invertible, $D_A f$ is also onto. Therefore, f has constant rank $\dim \mathrm{Sym}_n(\mathbb{R})$.

It follows that $O(p,q)$ is a Lie group as multiplication $m : \mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ and inversion $i : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ are smooth maps and hence restrict to smooth maps on the regular submanifold $O(p,q) \subset \mathrm{GL}(n,\mathbb{R})$.

Every symmetric matrix is uniquely determined by its entries above and on the diagonal such that

$$\dim \mathrm{Sym}_n(\mathbb{R}) = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}.$$

The constant rank theorem then yields

$$\begin{aligned} \dim O(p,q) = \dim f^{-1}(I_{p,q}) &= \dim \mathrm{GL}(n,\mathbb{R}) - \operatorname{rank} D_A f \\ &= n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}. \end{aligned}$$

Exercise 4. Let M be a smooth n-dimensional manifold and $p\in M$. Show that if (U,φ) is any chart at p with $\varphi(p) = 0$, then the map

$$\mathbb{R}^n \to T_pM, \quad v \mapsto (f \mapsto D_0(f\circ\varphi^{-1})(v))$$

is a vector space isomorphism.

Solution. We defined the tangent space T_pM at p as

$$T_pM = \{X_p \colon C^\infty(p) \to \mathbb{R}\colon \text{ linear forms that satisfy the Leibnitz rule }\}$$

where $C^\infty(p)$ is the set of germs of functions at p. We first note that the map ψ defined in the exercise does take values in T_pM. For $f,g\in C^\infty(p)$ and $\lambda\in\mathbb{R}$, have $\psi(v)(\lambda f+g) = D_0((\lambda f+g)\circ\varphi^{-1})(v) =$

$\lambda D_0(f \circ \varphi^{-1})(v) + D_0(g \circ \varphi^{-1})(v) = \lambda\psi(v)(f) + \psi(v)(g)$. Next, we check the Leibnitz rule, which follows from the product rule

$$\begin{aligned}
\psi(v)(f \cdot g) &= D_0(f \cdot g \circ \varphi^{-1})(v) \\
&= D_0((f \circ \varphi^{-1}) \cdot (g \circ \varphi^{-1}))(v) \\
&= D_0(f \circ \varphi^{-1})(v) \cdot (g \circ \varphi^{-1})(0) + (f \circ \varphi^{-1})(0) \cdot D_0(g \circ \varphi^{-1})(v) \\
&= \psi(v)(f) \cdot g(p) + f(p) \cdot \psi(v)(g).
\end{aligned}$$

We check that ψ is a linear map, let $v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}
\psi(\lambda v + w) &= f \mapsto D_0(f \circ \varphi^{-1})(\lambda v + w) \\
&= f \mapsto \lambda D_0(f \circ \varphi^{-1})(v) + D_0(f \circ \varphi^{-1})(w) \\
&= \lambda\psi(v) + \psi(w).
\end{aligned}$$

We give an explicit inverse

$$\begin{aligned}
\theta\colon T_pM &\to \mathbb{R}^n \\
X_p &\mapsto (X_p(f_1), \ldots, X_p(f_n))
\end{aligned}$$

where $f_i\colon M \to \mathbb{R}$ are defined as $f_i = \pi_i \circ \varphi$ with $\pi(v_1, \ldots, v_n) = v_i$. We check that $\theta(\psi(v)) = v$ for all $v \in \mathbb{R}^n$.

$$\begin{aligned}
\theta(\psi(v)) &= \big(D_0(f_1 \circ \varphi^{-1})(v), \ldots, D_0(f_n \circ \varphi^{-1})(v)\big) \\
&= (D_0(\pi_1)(v), \ldots, D_0(\pi_n)(v)) \\
&= (v_1, \ldots, v_n)\,.
\end{aligned}$$

For the other direction, we first describe some $f \in C^\infty(p)$. We consider the Taylor series at 0 of $f \circ \varphi^{-1}\colon \mathbb{R}^n \to \mathbb{R}$. The statement of Taylor's theorem for multivariate functions is that for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$f \circ \varphi^{-1}(x) = c + \sum_i \lambda_i \cdot x_i + \sum_{i,j} \mu_{ij}(x) \cdot x_i \cdot x_j$$

where

$$c = f \circ \varphi^{-1}(0) = f(p) \in \mathbb{R}, \quad \lambda_i = \left.\frac{\partial}{\partial x_i}\right|_0 f \circ \varphi^{-1} \in \mathbb{R}$$

and for $x \in \varphi(U)$

$$\mu_{ij}(x) = \left.\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right|_0 f \circ \varphi^{-1} + h_{ij}(x) \quad \text{with} \quad \lim_{x \to 0} h_{ij}(x) = 0.$$

In terms of functions, we have $f \circ \varphi^{-1} = c + \sum \lambda_i \cdot \pi_i + \sum \mu_{ij} \cdot \pi_i \cdot \pi_j$ and $f = c + \sum \lambda_i \cdot \pi_i \circ \varphi + \sum (\mu_{ij} \circ \varphi) \cdot (\pi_i \circ \varphi) \cdot (\pi_j \circ \varphi)$. Applying a tangent vector X_p to f gives

$$\begin{aligned}
X_p(f) &= X_p(c) + \sum_i \lambda_i X_p(\pi_i \circ \varphi) + \sum_{ij} X_p((\mu_{ij} \circ \varphi) \cdot (\pi_i \circ \varphi)) \cdot \pi_j(\varphi(p)) + \mu_{ij}(\varphi(p)) \cdot \pi_i(\varphi(p)) \cdot X_p(\pi_j \circ \varphi) \\
&= X_p(c) + \sum_i \lambda_i X_p(\pi_i \circ \varphi)
\end{aligned}$$

using the Leibnitz rule and the fact that $\pi_i \circ \varphi(p) = \pi_i(0) = 0$. For the constant term, we can use the Leibnitz rule again to obtain $X_p(c) = X_p(1 \cdot c) = X_p(1) \cdot c + 1 \cdot X_p(c)$, so $X(1) \cdot c = 0$. Thus $c = 0$ (hence $X_p(c) = 0$), or $X_p(1) = 0$ in which case also $X_p(c) = cX_p(1) = 0$. Let us now calculate $\psi(\theta(X_p))$

$$\begin{aligned}\psi(\theta(X_p))(f) &= D_0(f \circ \varphi^{-1}) \begin{pmatrix} X_p(f_1) \\ \vdots \\ X_p(f_n) \end{pmatrix} \\ &= \sum_i \left.\frac{\partial}{\partial x_i}\right|_0 (f \circ \varphi^{-1}) \cdot X_p(f_i) \\ &= \sum_i \lambda_i X_p(\pi_i \circ \varphi) = X_p(f).\end{aligned}$$

We have shown that θ is an inverse of ψ, hence $\mathbb{R}^n \cong T_pM$.