Solutions to Exercise Sheet 4

Exercise 1. We consider the determinant function $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$. Show that its differential at the identity matrix I is the trace function

$$D_I \det = \operatorname{tr}.$$

This calculation was used in the lecture to show that the Lie algebra of $\mathrm{SL}(n,\mathbb{R})$ is the set of traceless matrices.

Solution. Let $A \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned}
D_I\det(A) &= \frac{d}{dt}\Big|_{t=0} \begin{vmatrix} 1+ta_{1,1} & ta_{1,2} & \cdots & ta_{1,n} \\ ta_{2,1} & 1+ta_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,1} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \\
&\overset{(*)}{=} \frac{d}{dt}\Big|_{t=0} \left((1+ta_{1,1}) \begin{vmatrix} 1+ta_{2,2} & ta_{2,3} & \cdots & ta_{2,n} \\ ta_{3,2} & 1+ta_{3,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,2} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \right) \\
&\quad + \sum_{j=2}^{n} (-1)^{j+1} \frac{d}{dt}\Big|_{t=0} \left(ta_{2,j} \begin{vmatrix} ta_{1,2} & \cdots & ta_{1,n} \\ & * & \end{vmatrix} \right) \\
&\overset{(**)}{=} \left(a_{1,1} \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} + \frac{d}{dt}\Big|_{t=0} \begin{vmatrix} 1+ta_{2,2} & ta_{2,3} & \cdots & ta_{2,n} \\ ta_{3,2} & 1+ta_{3,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,2} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \right) \\
&\quad + \sum_{j=2}^{n} (-1)^{j+1} \left(a_{2,j} \begin{vmatrix} 0\cdot a_{1,2} & \cdots & 0\cdot a_{1,n} \\ & * & \end{vmatrix} + 0 \cdot * \right) \\
&= a_{1,1} + \frac{d}{dt}\Big|_{t=0} \det(I_{2\times n,2\times n} + tA_{2\times n,2\times n}) \\
&= \cdots = a_{1,1} + \cdots + a_{n,n} = \operatorname{tr}(A),
\end{aligned}$$

where we have developed the first column in $(*)$ and applied the product rule in $(**)$.

Exercise 2 (Some Lie Algebras)**.** (a) Let M, N be smooth manifolds and let $f : M \to N$ be a smooth map of constant rank r. By the constant rank theorem we know that the level set

$L = f^{-1}(q)$ is a regular submanifold of M of dimension $\dim M - r$ for every $q \in N$. Show that one may canonically identify

$$T_pL \cong \ker d_p f$$

for every $p \in L = f^{-1}(q)$.

Hint: Describe elements in T_pL as $\gamma'(0)$ for a smooth path $\gamma \colon (-\varepsilon, \varepsilon) \to M$.

Solution. Since $L = f^{-1}(q)$ is a regular submanifold of M we may think of the tangent space T_pL as a subspace of the tangent space T_pM. We will first show that $T_pL \subseteq \ker d_p f$. Let $v \in T_pL$ and let $\gamma : (-\varepsilon, \varepsilon) \to L = f^{-1}(q)$ be a smooth curve in L such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then $f(\gamma(t)) = q$ for all $t \in (-\varepsilon, \varepsilon)$, i.e. $f \circ \gamma$ is the constant curve. It follows that

$$d_pf(v) = d_{\gamma(0)}f(\gamma'(0)) = \frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) = 0.$$

In particular, $v \in \ker d_p f$ as claimed.

Finally, note that $\ker d_p f$ is a subspace of T_pM of dimension

$$\dim \ker d_p f = \dim T_pM - \operatorname{rank} d_pf = \dim M - r = \dim L = \dim T_pL.$$

Therefore T_pL is a linear subspace of $\ker d_p f$ of maximal dimension such that $T_pL = \ker d_p f$.

(b) Use part a) to compute the Lie algebras of the following Lie groups: $\mathrm{O}(n, \mathbb{R})$, $\mathrm{SO}(n, \mathbb{R})$, $\mathrm{O}(p, q)$, $B(n)$ the group of real invertible upper triangular matrices and $N(n)$ the subgroup of $B(n)$ with only ones on the diagonal.

Solution. Note that all of the listed Lie groups are subgroups of $\mathrm{GL}(n, \mathbb{R})$ that are also regular submanifolds. In particular the inclusion maps yield injective Lie algebra homomorphisms. This implies that the corresponding Lie algebras can be canonically identified with Lie subalgebras of $\mathfrak{gl}_n\mathbb{R}$. Hence the Lie bracket will be given by the ambient Lie bracket $[\cdot, \cdot]$ of $\mathfrak{gl}_n\mathbb{R}$. Identifying $\mathfrak{gl}_n\mathbb{R} \cong T_I \,\mathrm{GL}(n, \mathbb{R}) \cong \mathbb{R}^{n \times n}$ the Lie bracket is given by the commutator

$$[A, B] = AB - BA$$

as was proved in the lecture.

(i) $O(n, \mathbb{R})$: Consider the function $f_1 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n \times n}$ given by

$$f_1(A) = A^T A$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$. It is easy to check that f_1 has constant rank and that

$$O(n) = f_1^{-1}(I).$$

By part a)

$$\mathfrak{o}(n) := \mathrm{Lie}(O(n)) \cong T_I O(n) \cong \ker d_I f_1 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n \times n} \cong T_I \,\mathrm{GL}(n, \mathbb{R})$. We compute

$$\begin{aligned} d_I f_1(X) &= \frac{d}{dt}\bigg|_{t=0} (I + tX)^t (I + tX) \\ &= X^t + X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t + X = 0\}.$$

(ii) $SO(n, \mathbb{R})$. We notice that for $g \in O(n, \mathbb{R})$, $1 = \det(\mathrm{Id}) = \det(gg^\top) = \det(g)^2$, so $\det(g) \in \{\pm 1\}$. The determinant $\det\colon O(n, \mathbb{R}) \to \mathbb{R}$ is a continuous function and $\det(\mathrm{Id}) = 1$, so the connected component of the identity $O(n, \mathbb{R})^\circ \subseteq O(n, \mathbb{R})$ is a subset of $SL(n, \mathbb{R})$. Since the Lie algebra is defined locally as the tangent space at the identity, $\mathrm{Lie}(SO(n, \mathbb{R})) = \mathrm{Lie}(O(n, \mathbb{R})) = \mathfrak{o}(n, \mathbb{R})$, see part (i).

(iii) $O(p, q)$: Consider the function $f_2 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_2(A) = A^T I_{p,q} A$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$, where

$$I_{p,q} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p\text{-times}}, \underbrace{-1, \ldots, -1}_{q\text{-times}}).$$

It is easy to check that f_2 has constant rank and that

$$O(p, q) = f_2^{-1}(I_{p,q}).$$

By part a)

$$\mathfrak{o}(p, q) := \mathrm{Lie}(O(p, q)) \cong T_I O(p, q) \cong \ker d_I f_2 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I \,\mathrm{GL}(n, \mathbb{R})$. We compute

$$\begin{aligned} d_I f_2(X) &= \frac{d}{dt}\bigg|_{t=0} (I + tX)^t I_{p,q}(I + tX) \\ &= X^t I_{p,q} + I_{p,q} X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{o}(p, q) = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t I_{p,q} + I_{p,q} X = 0\}.$$

(iv) $B(n)$: Consider the function $f_5 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_5(A) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ A_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_{n1} & \cdots & A_{n,n-1} & 0 \end{pmatrix}$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$. It is easy to check that f_5 has constant rank and that

$$B(n) = f_5^{-1}(0).$$

By part a)

$$\mathfrak{b}(n) := \mathrm{Lie}(B(n)) \cong T_I B(n) \cong \ker d_I f_5 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I \,\mathrm{GL}(n,\mathbb{R})$. We compute

$$d_I f_5(X) = \frac{d}{dt}\bigg|_{t=0} f_5(I+tX)$$
$$= \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ X_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ X_{n1} & \cdots & X_{n,n-1} & 0 \end{pmatrix}.$$

Therefore

$$\mathfrak{b}(n) = \left\{ \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ & \ddots & \vdots \\ 0 & & X_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n} \right\}.$$

(v) $N(n)$: Consider the function $f_6 : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_6(A) = \begin{pmatrix} X_{11} & & 0 \\ \vdots & \ddots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

for every $A \in \mathrm{GL}(n,\mathbb{R})$. It is easy to check that f_6 has constant rank and that

$$N(n) = f_6^{-1}(I).$$

By part a)

$$\mathfrak{n}(n) := \mathrm{Lie}(N(n)) \cong T_I N(n) \cong \ker d_I f_6 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I \,\mathrm{GL}(n,\mathbb{R})$. We compute

$$d_I f_6(X) = \frac{d}{dt}\bigg|_{t=0} f_6(I+tX)$$
$$= \begin{pmatrix} X_{11} & & 0 \\ \vdots & \ddots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}.$$

Therefore

$$\mathfrak{n}(n) = \left\{ \begin{pmatrix} 0 & * & * \\ \vdots & \ddots & * \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n} \right\}.$$

Exercise 3 (One- and two-dimensional Lie Algebras)**.** Classify the one- and two-dimensional real Lie algebras up to Lie algebra isomorphism and realize them as Lie subalgebras of some $\mathfrak{gl}_n\mathbb{R} = \mathfrak{gl}(\mathbb{R}^n)$.

Hint: In dimension two one can show that if the Lie algebra is non-abelian then there is a basis X, Y such that $[X,Y] = Y$.

Solution. Let $(\mathfrak{a}, [\cdot,\cdot])$ be a real Lie algebra.

We will first deal with the one-dimensional case. Suppose $\dim \mathfrak{a} = 1$ and let X be a basis vector for $\mathfrak{a}$. Due to the anti-symmetry of the Lie bracket we have

$$[X, X] = -[X, X] = 0,$$

i.e. every one-dimensional Lie algebra is abelian. We claim that the linear map $\varphi : (\mathfrak{a}, [\cdot,\cdot]) \to (\mathbb{R}, [\cdot,\cdot])$ given by $\varphi(X) = 1$ is a Lie algebra isomorphism where the Lie bracket on $\mathbb{R}$ vanishes everywhere. Clearly, φ is an isomorphism of vector spaces and

$$[\varphi(X), \varphi(X)] = 0 = \varphi(\underbrace{[X, X]}_{=0})$$

such that φ is indeed a Lie algebra isomorphism.

In order to realize $\mathfrak{a}$ as a Lie subalgebra of some $\mathfrak{gl}_n\mathbb{R}$ we need to find a one-dimensional subalgebra of some $\mathfrak{gl}_n\mathbb{R}$ on which the commutator $[\cdot,\cdot]$ in $\mathfrak{gl}_n\mathbb{R}$ vanishes. Consider

$$\mathfrak{b} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\} \subseteq \mathfrak{gl}_2\mathbb{R}.$$

Clearly, $\mathfrak{b}$ is a linear subspace of $\mathfrak{gl}_2\mathbb{R}$. Further, note that

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x \cdot y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

for all $x, y \in \mathbb{R}$, such that $[X, Y] = 0$ for all $X, Y \in \mathfrak{b}$. Therefore the vector space isomorphism $\psi : \mathbb{R} \to \mathfrak{b}$ given by

$$\psi(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

is also a Lie algebra isomorphism. Thus, $\psi \circ \varphi : \mathfrak{a} \hookrightarrow \mathfrak{gl}_2(\mathbb{R})$ realizes $\mathfrak{a}$ as a Lie subalgebra of $\mathfrak{gl}_2\mathbb{R}$.

Suppose $\dim \mathfrak{a} = 2$ and let $\{X, Y\}$ be a basis of $\mathfrak{a}$. Suppose $\mathfrak{a}$ is abelian, i.e. $[X, Y] = 0$. Consider

$$\mathfrak{c} := \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R} \right\} \subset \mathfrak{gl}_2\mathbb{R}$$

and the vector space isomorphism $\varphi : \mathfrak{a} \to \mathfrak{c}$ given by

$$\varphi(X) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: E_{11}, \quad \varphi(Y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: E_{22}.$$

Note that

$$E_{11} \cdot E_{22} = 0 = E_{22} \cdot E_{11},$$

such that

$$\varphi([X, Y]) = \varphi(0) = 0 = [E_{11}, E_{22}] = [\varphi(X), \varphi(Y)].$$

Therefore, $\varphi : \mathfrak{a} \to \mathfrak{c} \subset \mathfrak{gl}_2\mathbb{R}$ is a Lie algebra isomorphism. This realizes $\mathfrak{a}$ as the subalgebra $\mathfrak{c}$ of $\mathfrak{gl}_2\mathbb{R}$ and shows that every real abelian Lie algebra is isomorphic to $\mathfrak{c}$.

Finally, suppose that $\mathfrak{a}$ is non-abelian such that

$$[X,Y] = \alpha X + \beta Y \neq 0 \tag{$\star$}$$

for some $\alpha, \beta \in \mathbb{R}$. By ($\star$) not both α and β are zero such that

$$\beta\lambda - \alpha\mu = 1$$

for some $\lambda, \mu \in \mathbb{R}$. Define

$$X' := \lambda X + \mu Y, \quad Y' := \alpha X + \beta Y = [X,Y].$$

Observe that the base change from $\{X, Y\}$ to $\{X', Y'\}$ is given by the matrix

$$\begin{pmatrix} \lambda & \alpha \\ \mu & \beta \end{pmatrix}$$

with determinant $\lambda\beta - \alpha\mu = 1$ such that $\{X', Y'\}$ is again a basis of $\mathfrak{a}$. Further,

$$\begin{aligned} [X', Y'] &= [\lambda X + \mu Y, \alpha X + \beta Y] \\ &= \lambda\beta[X,Y] + \mu\alpha[Y,X] \\ &= (\beta\lambda - \alpha\mu)[X,Y] \\ &= Y'. \end{aligned}$$

Consider the vector subspace $\mathfrak{d} \subset \mathfrak{gl}_2\mathbb{R}$ generated by the matrices

$$A := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In fact, $\mathfrak{d}$ is a Lie subalgebra:

$$\begin{aligned} [A, C] &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = C. \end{aligned}$$

This computation also shows that the linear map $\varphi : \mathfrak{a} \to \mathfrak{d}$ given by

$$\varphi(X') = A, \quad \varphi(Y') = C$$

is a Lie algebra isomorphism (it is easily seen to be an isomorphism of vector spaces). Therefore, $\mathfrak{a}$ can be realized as the subalgebra $\mathfrak{d}$ of $\mathfrak{gl}_2\mathbb{R}$. This also proves that any real, non-abelian Lie algebra $\mathfrak{a}$ is isomorphic to $\mathfrak{d}$.

<u>Remark:</u> Notice that the map $\Phi : \mathfrak{gl}_2\mathbb{R} \hookrightarrow \mathfrak{gl}_n\mathbb{R}$ given by

$$\Phi(A) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$$

is an injective Lie algebra homomorphism such that the discussed realizations of $\mathfrak{a}$ as subalgebras of $\mathfrak{gl}_2\mathbb{R}$ also amount to realizations of $\mathfrak{a}$ in any $\mathfrak{gl}_n\mathbb{R}$.

Exercise 4. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Show that the Lie algebra of $G \times H$ can be identified with $\mathfrak{g} \times \mathfrak{h}$ with the bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2]_{\mathfrak{g}}, [y_1, y_2]_{\mathfrak{h}}).$$

Solution. We are going to denote by $\mathfrak{X}(M) = \mathrm{Vect}^\infty(M)$ the set of vector fields over a generic manifold M.

We are going to denote by

$$i_G : G \to G \times H, \quad i_G(g) := (g, e)$$

and similarly

$$i_H : H \to G \times H, \quad i_H(h) := (e, h).$$

In the same way, the differential of both maps induces inclusions

$$D_e i_G : T_eG \to T_eG \times T_eH, \quad D_e i_G(u) := (u, 0)$$

and

$$D_e i_H : T_eH \to T_eG \times T_eH, \quad D_e i_G(v) := (0, v).$$

Recall that $T_eG \times T_eH$ is canonically isomorphic to $T_eG \oplus T_eH$ as $\mathbb{R}$-vector spaces via the map which sends (u, v) to $u + v$, for every $u \in T_eG$ and every $v \in T_eH$. (In this way we get $D_e i_G$ is simply the inclusion of T_eG into $T_eG \oplus T_eH$ and the same for $D_e i_H$). This means that every element w in $T_{(e,e)}(G \times H)$ can be written uniquely as $w = u + v$, where $u \in T_eG$ and $v \in T_eH$, or equivalently we can identify $T_{(e,e)}(G \times H)$ with $T_eG \oplus T_eH$.

From the lecture, we know that there is a bijection between left-invariant vector fields on G (resp. H) and vectors of the tangent space T_eG (resp. T_eH) and the isomorphism is given by

$$L_G : T_eG \to \mathfrak{X}(G)^G, \quad L_G(u) := u^L$$

where the vector field u^L is defined at the point $g \in G$ as $u^L_g := D_e L_g(u)$.

It should be clear that we have the following commutative diagram

$$\begin{tikzcd}
T_eG \oplus T_eH \arrow[r, "\cong"] \arrow[d, "L_G \oplus L_H"] & T_{(e,e)}(G \times H) \arrow[d, "L_{G \times H}"] \\
\mathfrak{X}(G)^G \oplus \mathfrak{X}(H)^H \arrow[r, "\cong"] & \mathfrak{X}(G \times H)^{G \times H}
\end{tikzcd}$$

The diagram above is telling us that every $(G \times H)$-left-invariant vector field $Z = w^L$, where $w \in T_{(e,e)}(G \times H)$, can be uniquely written as $Z = X + Y$, where $X = u^L$ (resp. $Y = v^L$) where

$u \in T_eG$ (resp. $v \in T_eH$). Here the left-invariance property has to be understood in $G \times H$ (that means that both u^L and v^L are $G \times H$ left-invariant).

Take now $Z_1, Z_2 \in \mathfrak{X}(G \times H)^{G\times H}$ of the form $Z_i = w_i^L$, where $w_i \in T_{(e,e)}(G \times H)$ for $i = 1, 2$. By what we have said so far there exist unique $u_i \in T_eG$ and $v_i \in T_eH$ such that $w_i^L = u_i^L + v_i^L$, for $i = 1, 2$. It holds

$$\begin{aligned}[Z_1, Z_2] =&[w_1^L, w_2^L] = [u_1^L + v_1^L, u_2^L + v_2^L] = \\ =&[u_1^L, u_2^L] + [u_1^L, v_2^L] + [v_1^L, u_2^L] + [v_1^L, v_2^L].\end{aligned}$$

It is immediate to verify that for every $[u^L, v^L] = 0$ for any $u \in T_eG$ and $v \in T_eH$, hence we get

$$[w_1^L, w_2^L] = [u_1^L, u_2^L] + [v_1^L, v_2^L],$$

which is exactly the Lie algebra structure given on the product, and we are done.

Exercise 5 (Surjectivity of the Matrix Exponential)**.** Let $\mathrm{Exp} : \mathfrak{gl}(n, \mathbb{R}) \cong \mathbb{R}^{n\times n} \to \mathrm{GL}(n, \mathbb{R})$ be the matrix exponential map given by the power series

$$\mathrm{Exp}(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Consider the Lie subgroup of upper triangular matrices with 1's on the diagonal $N(n) < \mathrm{GL}(n, \mathbb{R})$ with its Lie algebra $\mathfrak{n}(n) < \mathfrak{gl}(n, \mathbb{R})$ of strictly upper triangular matrices; cf. Exercise 1.

Show that $\mathrm{Exp}|_{\mathfrak{n}(n)} : \mathfrak{n}(n) \to N(n)$ is surjective.

<u>Hint:</u> Consider the partially defined matrix logarithm $\mathrm{Log} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$\mathrm{Log}(I + A) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n}.$$

Try to give answers to the following questions and then conclude:

What is its radius of convergence r about I? Why is it a right-inverse of Exp on the ball $B_r(I)$ of radius r about I? Why is there no problem for matrices that are in $N(n)$ but not in $B_r(I)$?

In order to answer the last question prove that $A^n = 0$ for all $A \in \mathfrak{n}(n)$.

Solution. Note that

$$r = \lim_{n\to\infty} \left| \frac{(-1)^{n-1}}{n} \cdot \frac{n+1}{(-1)^n} \right| = 1$$

such that the power series $\mathrm{Log}(I + A)$ converges absolutely for every $A \in \mathbb{R}^{n\times n}$ with $\|A\| < 1$ as in the complex case.

For all complex numbers $z \in \mathbb{C}$ with $|z| < 1$ we have

$$e^{\log(1+z)} = 1 + z. \tag{1}$$

Recall that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C}$$

and

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \qquad \forall z \in B_1(0) \subset \mathbb{C}.$$

Writing the composition $e^{\log(1+z)}$ as a power series we obtain

$$e^{\log(1+z)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \right)^k = \sum_{k=0}^{\infty} d_k z^k$$

for all $z \in B_1(0) \subset \mathbb{C}$ for some $d_k \in \mathbb{R}$, where one uses succesively the Cauchy product rule for power series to compute the power series representation of $\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}\right)^k$ and then uses the fact that the series converges absolutely for $|z| < 1$ to reorder it and to obtain the coefficients for each z^k.

Comparing coefficients in (1) then yields that $d_0 = d_1 = 1$ and $d_k = 0$ for all $k > 1$.

Let us now write $\operatorname{Exp}(\operatorname{Log}(I + A))$ as well as a power series

$$\operatorname{Exp}(\operatorname{Log}(I + A)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n} \right)^k = \sum_{k=0}^{\infty} d_k A^k$$

for all $z \in B_1(0) \subset \mathbb{C}$, where one uses succesively the Cauchy product rule for power series to compute the power series representation of $\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}\right)^k$ and then uses the fact that the series converges absolutely for $|z| < 1$ to reorder it and to obtain the coefficients for each z^k as above.

Observe that the coefficients $d_k \in \mathbb{R}$ are the same as in the complex case! This is due to the fact that they arise from the same computation with power series (Cauchy product rule and reordering accordingly). Hence, $d_0 = d_1 = 1$ and $d_k = 0$ for all $k > 1$ such that

$$\operatorname{Exp}(\operatorname{Log}(I + A)) = I + A$$

for every $A \in \mathbb{R}^{n \times n}$ with $\|A\| < 1$. [1]

Finally, observe that every $X \in N(n)$ can be written as $X = I + A$ where $A \in \mathfrak{n}(n)$. Furthermore, since A is strictly upper triangular it maps

$$A|_{V_i} : V_i \to V_{i-1}$$

[1] The reasoning applied here can be generalized. In fact, there are theorems that relate identities of complex power series to identities of power series in Banach algebras; see e.g. Königsberger: ,,Analysis 2", ch. 1.6 and Königsberger: ,,Analysis 2", Exercise 18, p. 44

where $V_i = \operatorname{span}\{e_1, \dots, e_i\}$, $V_0 = \{0\}$ for every $i = 1, \dots, n$. In particular,

$$A^n : \mathbb{R}^n = V_n \to V_{n-1} \to \cdots \to V_0 = \{0\}$$

such that $A^n = 0$.

That means that for every $A \in \mathfrak{n}(n)$ the power series $\operatorname{Log}(I + A)$ is actually a polynomial in A taking values in $\mathfrak{n}(n)$:

$$\operatorname{Log}(I + A) = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{A^k}{k} \in \mathfrak{n}(n).$$

Because $\operatorname{Log}(I + A)$ is again in $\mathfrak{n}(n)$ also $\operatorname{Exp}(\operatorname{Log}(I + A))$ becomes a polynomial p in A:

$$\operatorname{Exp}(\operatorname{Log}(I + A)) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\left(\sum_{l=1}^{n-1} (-1)^{l-1} \frac{A^l}{l} \right)^k}_{=0, \text{ if } k \geq n} = \sum_{k=0}^{n-1} \frac{1}{k!} \left(\sum_{l=1}^{n-1} (-1)^{l-1} \frac{A^l}{l} \right)^k =: p(A)$$

Now observe that $\|tA\| < 1$ for all $t \in I_A := (-\|A\|^{-1}, \|A\|^{-1}) \subset \mathbb{R}$. Hence,

$$p(tA) = \operatorname{Exp}(\operatorname{Log}(I + tA)) = I + tA$$

for all $t \in I_A$. The left-hand-side and the right-hand-side are both polynomials in t which coincide on an open subset of $\mathbb{R}$. Thus they have to coincide everywhere; in particular

$$\operatorname{Exp}(\operatorname{Log}(I + A)) = I + A$$

for $t = 1$. This shows that $\operatorname{Log}|_{N(n)}$ is a well-defined right-inverse of $\operatorname{Exp}|_{\mathfrak{n}(n)}$.