Solutions to Exercise Sheet 5

Exercise 1 (Discrete Subgroups of $\mathbb{R}^n$)**.** Let $D < \mathbb{R}^n$ be a discrete subgroup. Show that there are $x_1, \ldots, x_k \in D$ such that

- $x_1, \ldots, x_k$ are linearly independent over $\mathbb{R}$, and
- $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$, i.e. $x_1, \ldots, x_k$ generate D as a $\mathbb{Z}$-submodule of $\mathbb{R}^n$.

Solution. We will prove this by induction on the dimension n.

Let $n = 1$ and let $D < \mathbb{R}$ be a discrete subgroup. Without loss of generality we may assume that $D \neq \{0\}$. Since D is discrete there is $x_1 \in D \setminus \{0\}$ such that $|x_1| = \min\{|x| : x \in D \setminus \{0\}\}$. We claim that $D = \mathbb{Z}x_1$. Suppose there is $y \in D \setminus \mathbb{Z}x_1$. Then there is $k \in \mathbb{Z}$ such that

$$k \cdot x_1 < y < (k+1) \cdot x_1.$$

It follows that $y - k \cdot x_1 \in D$ and $|y - k \cdot x_1| < |x_1|$ which contradicts the minimality of x_1. This shows that $D = \mathbb{Z}x_1$ and finishes the proof of the base case $n = 1$.

Let $n \in \mathbb{N}$ and assume the statement holds for all discrete subgroups of $\mathbb{R}^{n-1}$. Let $D < \mathbb{R}^n$ be a discrete subgroup. Without loss of generality we may assume that $D \neq \{0\}$. There is $x_1 \in D \setminus \{0\}$ such that $\|x_1\| = \min\{\|x\| : x \in D \setminus \{0\}\}$. Consider the quotient $\mathbb{R}^n/\mathbb{R}x_1 \cong \mathbb{R}^{n-1}$ and the projection

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{R} \cdot x_1 \cong \mathbb{R}^{n-1}$$

onto it.

We claim that $D' = \pi(D) < \mathbb{R}^{n-1}$ is a discrete subgroup. We will see this by showing that $V' := \pi(B_r(0))$ is an open neighborhood of $0 \in D'$ such that $V' \cap D' = \{0\}$ where $r := \inf\{\|t \cdot x_1 - y\| : t \in \mathbb{R}, y \in D \setminus \mathbb{Z}x_1\}$.

First of all, we need to see that r is in fact positive. In order to prove this let us verify that

$$r = \inf\{\|t \cdot x_1 - y\| : t \in \mathbb{R}, y \in D \setminus \mathbb{Z}x_1\} = \inf\{\|t \cdot x_1 - y\| : t \in [0,1], y \in D \setminus \mathbb{Z}x_1\}.$$

Clearly, the left-hand-side is less than or equal to the right-hand-side. On the other hand, if $R \geq 0$ such that there are $t \in \mathbb{R}$ and $y \in D \setminus \mathbb{Z}x_1$ satisfying $R \geq \|t \cdot x_1 - y\|$ then also

$$R \geq \|t \cdot x_1 - y\| = \|(t - \lfloor t \rfloor)x_1 - (y - \lfloor t \rfloor\, x_1)\|;$$

whence there are $s := t - \lfloor t \rfloor \in [0,1]$ and $w := (y - \lfloor t \rfloor\, x_1) \in D \setminus \mathbb{Z}x_1$ such that $R \geq \|s \cdot x_1 - w\|$. Therefore, the right-hand-side is also less than or equal to the left-hand-side such that they must be equal. Because $\{t \cdot x_1 : t \in [0,1]\} \subset \mathbb{R}^n$ is compact and $D \setminus \mathbb{Z}x_1$ is discrete the infimum on the right-hand-side is in fact a minimum. It is attained at some $t_0 \cdot x_1$ and $y_0 \in D \setminus \mathbb{Z}x_1$. If $r = \|t_0 \cdot x_1 - y_0\| = 0$ then $y_0 = t_0 x_1$ and $t_0 \in (0,1)$ because $y_0 \notin \mathbb{Z}x_1$. But then $\|y_0\| = t_0\|x_1\| < \|x_1\|$ which contradicts the minimality of $\|x_1\|$; whence $r > 0$.

Clearly, $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is an open map such that $V' = \pi(B_r(0))$ is an open neighborhood of $0 \in \mathbb{R}^{n-1}$. Now, let $x' \in D' \cap V'$, i.e. $x' = \pi(u) = \pi(y)$ for some $u \in B_r(0)$, $y \in D$. Then $y - u \in \mathbb{R}x_1$, i.e. $y - u = t \cdot x_1$ for some $t \in \mathbb{R}$. This implies that

$$\|y - t \cdot x_1\| = \|u\| < r = \inf\{\|y - t \cdot x_1\| : t \in \mathbb{R}, y \in D \setminus \mathbb{Z}x_1\}.$$

We deduce that $y \in \mathbb{Z}x_1 \subset \mathbb{R}x_1$; whence $x' = \pi(y) = 0$ and $V' \cap D' = \{0\}$. Therefore, 0 is an isolated point in D' such that D' is a discrete subgroup of $\mathbb{R}^{n-1}$ as claimed.

By the induction hypothesis there are $x'_2, \ldots, x'_k \in D' < \mathbb{R}^{n-1}$ which are linearly independent over $\mathbb{R}$ and generate D' as a $\mathbb{Z}$-submodule, i.e. $D' = \mathbb{Z}x'_2 \oplus \cdots \oplus \mathbb{Z}x'_k$. We choose for every x'_i a preimage $x_i \in \pi^{-1}(x'_i) \cap D$. These $x_1, x_2, \ldots, x_k \in D$ are linearly independent over $\mathbb{R}$ and satisfy $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$. Indeed, let $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k = 0. \tag{1}$$

Then

$$\begin{aligned} 0 &= \pi(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) \\ &= \underbrace{\lambda_1 \pi(x_1)}_{=0} + \lambda_2 \pi(x_2) + \cdots + \lambda_k \pi(x_k) \\ &= \lambda_2 x'_2 + \cdots + \lambda_k x'_k. \end{aligned}$$

Because $x'_2, \ldots, x'_k$ are linearly independent, $\lambda'_2 = \ldots = \lambda'_k = 0$. By (1), $\lambda_1 x_1 = 0$. Finally, since $x_1 \neq 0$ also $\lambda_1 = 0$.

In order to see that $x_1, \ldots, x_k$ generate D as a $\mathbb{Z}$-module, let $y \in D$. Then

$$\pi(y) = a_2 x'_2 + \cdots + a_k x'_k = a_2 \pi(x_2) + \cdots + a_k \pi(x_k)$$

for some $a_2, \ldots, a_k \in \mathbb{Z}$ since $x'_2, \ldots, x'_k$ generate D' as a $\mathbb{Z}$-module. Considering $y' = a_2 x_2 + \cdots + a_k x_k \in D$ we obtain

$$\pi(y') = \pi(a_2 x_2 + \cdots + a_k x_k) = a_2 \pi(x_2) + \cdots + a_k \pi(x_k) = \pi(y)$$

by linearity such that $y - y' \in D \cap \ker\pi = D \cap \mathbb{R}x_1$.

We claim that $D \cap \ker\pi = \mathbb{Z}x_1$. It is immediate that $\mathbb{Z}x_1 \subseteq D \cap \ker\pi$. To see the other inclusion suppose that there is $t \cdot x_1 \in D$ for some $t \in \mathbb{R} \setminus \mathbb{Z}$. Then $w = (t - \lfloor t \rfloor) \cdot x_1 \in D \setminus \{0\}$ and

$$\|w\| = (t - \lfloor t \rfloor) \cdot \|x_1\| < \|x_1\|$$

in contradiction to the minimality of x_1.

Therefore, $y - y' \in \mathbb{Z}x_1$ and there exists $a_1 \in \mathbb{Z}$ such that

$$y = a_1 x_1 + y' = a_1 x_1 + a_2 x_2 + \cdots + a_k x_k.$$

Hence, $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$.

Exercise 2. Show that every connected abelian Lie group G is isomorphic as a Lie group to $\mathbb{T}^a \times \mathbb{R}^{n-a}$ for some $a \in \{0, \dots, n\}$, where $n = \dim G$ and $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$.

Solution. The Lie algebra of G is isomorphic to the abelian Lie algebra $\mathfrak{g} := \mathbb{R}^n$, which also happens to be the Lie algebra of the Lie group $H := \mathbb{R}^n$. By Theorem 3.89, the Lie algebra isomorphism $\mathrm{Lie}(H) \cong \mathrm{Lie}(G)$ lifts to a local Lie group isomorphism $U \to G$ for some open neighborhood $U \subseteq H$ of $0 \in H$. By the Local Isomorphism Theorem 2.37, since H is path-connected and simply connected, the local Lie group isomorphism extends to a continuous homomorphism $\varphi\colon H \to G$. We note that $\varphi(U)$ is a neighborhood of $e \in G$, and φ is surjective since

$$G = \bigcup_{n \in \mathbb{N}} \varphi(U)^n$$

by one of the first properties of topological groups. By the isomorphism theorem, we obtain $G \cong H/\ker(\varphi)$ as groups. We claim that $\ker(\varphi)$ is discrete, indeed $\ker(\varphi) \cap U = \{0\}$, whence $\{0\}$ is open. Since $\ker(\varphi)$ acts on H by homeomorphism, all singletons in $\ker(\varphi)$ are open, hence $\ker(\varphi)$ is discrete.

By Exercise 1 of this sheet, there are linearly independent $x_1, \dots, x_a$ such that $\ker(\varphi) = \bigoplus_i \mathbb{Z}x_i$. Let $V = \langle x_1, \dots, x_a \rangle$ and $H = \mathbb{R}^n = V \times V^\perp$.

$$G = H/\ker(\varphi) = V \times V^\perp/\ker(\varphi) = V/\ker(\varphi) \times V^\perp \cong \mathbb{R}^a/\mathbb{Z}^a \times \mathbb{R}^{n-a}$$

first of all as abstract groups, but also as Lie groups, since φ is a smooth cover since $\ker(\varphi)$ is discrete.

Exercise 3 (Easy Direction of Frobenius' Theorem)**.** Let M be a smooth manifold and let $\mathcal{D}$ be a distribution on M. Show that $\mathcal{D}$ is involutive if it is completely integrable.

Solution. Let $U \subset M$ be an open set and $\{X_1, \dots, X_n\}$ a local basis of $\mathcal{D}$ defined on U. Further, let $q \in U$ and suppose q is contained in an integral submanifold $\varphi : N \hookrightarrow M$ of $\mathcal{D}$ such that $d_p\varphi(T_pN) = \mathcal{D}_p$ for every $p \in N$, where $\varphi : N \hookrightarrow M$ is an injective immersion. Let $p \in \varphi^{-1}(q)$ and choose open neighborhoods $V' \subset N$ about p and $U' \subset U$ about q such that $\varphi|_{V'} : V' \to U'$ is a smooth embedding. By using a local slice chart it is easy to see that the vector fields $\{Y_1, \dots, Y_n\}$ defined via

$$d_{p'}\varphi(Y_i) = (X_i)_{\varphi(p')} \qquad \forall p' \in V' \, \forall i = 1, \dots, n \tag{$\star\star$}$$

are smooth vector fields on $V' \subset N$. Here we have used that $\{(X_1)_{\varphi(p')}, \dots, (X_n)_{\varphi(p')}\}$ is a basis of $\mathcal{D}_{\varphi(p')} = d_{p'}\varphi(T_{p'}N)$ and that the differential of $d_{p'}\varphi$ is injective for every $p' \in V'$. Note that $(\star\star)$ means that Y_i is φ-related to X_i for every $i = 1, \dots, n$. By exercise 1 also $[Y_i, Y_j]$ is φ-related to $[X_i, X_j]$, i.e.

$$[X_i, X_j]_{\varphi(p')} = d_{p'}\varphi[Y_i, Y_j]_{p'},$$

for every $i, j = 1, \dots, n$. Because $\{Y_1, \dots, Y_n\}$ are smooth vector fields on $V' \subset N$ also $[Y_i, Y_j]_{p'}$ is a smooth vector field on $V' \subset N$. This implies that $[X_i, X_j]_{\varphi(p')} \in d_{p'}\varphi(T_{p'}N) = \mathcal{D}_{\varphi(p')}$ for every $p' \in V'$; in particular $[X_i, X_j]_q \in \mathcal{D}_q$. Therefore $\mathcal{D}$ is involutive.

Exercise 4. (a) Show that every continuous group homomorphism from $\mathbb{R}$ to a Lie group is smooth.

Solution. This is Proposition 3.13 in Prof. Alessandra Iozzi's notes, proved on pages 86 and 87

(b) Show that every continuous group homomorphism between two Lie groups is smooth.

Solution. This is Theorem 3.14 in Prof. Alessandra Iozzi's notes, proved on pages 85 and 86 using part (a) of the exercise.

In fact, one can prove that every measurable homomorphism between two locally compact topological groups is continuous, hence every measurable homomorphism between two Lie groups is smooth.

Exercise 5. (Corollary 3.93(2)) Show that if two simply connected, connected Lie groups G_1, G_2 have isomorphic Lie algebra, then they are isomorphic.

Solution. Theorem 3.89 from the lecture states that since the Lie algebras are isomorphic $f\colon \mathfrak{g}_1 \cong \mathfrak{g}_2$, there is a local isomorphism $\varphi\colon U \to G_2$ for an open neighborhood $U \subseteq G_1$ of $e_1 \in G_1$. Now since G_1 is is simply connected (and connected, hence path-connected), there is a unique lift $\bar{\varphi}\colon G_1 \to G_2$, which is a continuous homomorphism (Theorem 2.37). Now we can apply Exercise 4(b) or use the fact that it is smooth around e_1 to make sure that $\bar{\varphi}$ is a smooth homomorphism with $D_{e_1}\bar{\varphi} = f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Similarly we can invert the role of G_1 and G_2 to obtain a smooth homomorphism $\bar{\psi}\colon G_2 \to G_1$ with $D_{e_2}\bar{\psi} = f^{-1}\colon \mathfrak{g}_2 \to \mathfrak{g}_1$.

We may assume that the neighborhoods $U \subseteq G_1$ and $V \subseteq G_2$ on which the local isomorphisms φ, ψ are defined, are given by $U = \exp(L)$ and $V = \exp(f(L))$ for an open subset $L \subseteq \mathfrak{g}_1$, so that $\varphi(U) = V$ and $\psi(V) = U$.

We now claim that $\bar{\varphi}$ is a covering, i.e. for every $g_2 \in G_2$ there is a neighborhood $V_2 \subseteq G_2$ of g_2 such that $\bar{\varphi}^{-1}(V_2)$ is a disjoint union of open sets $U_1 \subseteq G_1$ wich are homeomorphic to V_2.

For $g_2 \in G_2$ we take $g_2 V$ as the open neighborhood. We have

$$\bar{\varphi}^{-1}(g_2 V) = \cup_{g \in \bar{\varphi}^{-1}(g_2)} gU$$

so we just have to prove that the gU are disjoint: if there are $g, g' \in \bar{\varphi}^{-1}(g_2)$ with $g' \in gU$, then $g^{-1}g' \in U$, and so $\bar{\varphi}(g^{-1}g') = \bar{\varphi}(g)^{-1}\bar{\varphi}(g') = g_2^{-1}g_2 = e_2 \in \bar{\varphi}(U) = V$, hence $g^{-1}g' = e_1$ and $g = g'$.

Now a covering of a simply connected space is bijective, so $\bar{\varphi}\colon G_1 \to G_2$ is a bijective smooth group homomorhpism with invertable differential, so the inverse is also a smooth group homomorphism, so $G_1 \cong G_2$ are isomorphic Lie groups.