Solutions to exercise Sheet 6

Exercise 1 (The adjoint representation is smooth)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}(g)$ is smooth, where $\mathrm{Ad}(g) := D_e(\mathrm{int}(g))$.

Hint: Consider the map $\mathrm{int}(g)\colon G \to G$, $x \mapsto gxg^{-1}$. Use that $\exp_G$ is a local diffeomorphism to conclude that Ad is smooth near e. Then use left translation to show that Ad is smooth everywhere.

Solution. Consider the map $F\colon G \times G \to G$ defined by $F(g,h) := ghg^{-1}$. This is smooth, so its differential $DF\colon TG \times TG \to TG$ is smooth. Restrict in the second component to the submanifold $T_eG = \mathfrak{g}$. The zero vector field $0\colon G \to TG$ is a smooth map, thus the map

$$G \times \mathfrak{g} \to TG,\ (g,X) \mapsto DF(0(g),X)$$

is smooth as well. From the construction, we have

$$D_{(g,e)}F(0(g),X) = \tfrac{d}{dt}_{|t=0}F(g,\exp(tX)) = \tfrac{d}{dt}_{|t=0} g\exp(tX)g^{-1} = \mathrm{Ad}(g)(X) \in T_eG.$$

Thus the map $G \times \mathfrak{g} \to \mathfrak{g}$, $(g,X) \mapsto \mathrm{Ad}(g)(X)$ is smooth. If you choose a basis for $\mathfrak{g}$, say $\{X_i\}$ with dual basis $\{X_i^*\}$, then the entries of the matrix representing $\mathrm{Ad}(g)$ with respect to the basis $\{X_i\}$ is $X_i^*(\mathrm{Ad}(g)(X_j))$, so they depend smoothly on g, thus $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ is smooth.

Exercise 2 ($\mathrm{Z}(G) = \mathrm{Ker}(\mathrm{Ad})$)**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Use the fundamental relation that

$$g\exp(tX)g^{-1} = \exp(t\,\mathrm{Ad}_g(X))$$

for all $g \in G, t \in \mathbb{R}$ and $X \in \mathfrak{g}$ to prove the following.

(1) If G is connected, then the center $\mathrm{Z}(G)$ of G equals the kernel of the adjoint representation.

Solution. If $g \in \mathrm{Z}(G)$, then we have for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$ that

$$\exp(tX) = g\exp(tX)g^{-1} = \exp(t\,\mathrm{Ad}_g(X))$$

and while exp may not be injective on all of $\mathfrak{g}$, it is injective on an open neighborhood of 0, in particular there is a vector-space basis of $\mathfrak{g}$ contained in the open neighborhood of 0 such that $X = \mathrm{Ad}_g(X)$ for all elements of the basis. By linear extension, we then have that $\mathrm{Ad}_g = \mathrm{Id}$, so $g \in \mathrm{Ker}(\mathrm{Ad})$.

If on the otherhand we start with $g \in \mathrm{Ker}(\mathrm{Ad})$, we apply the same formula to see that g commutes with all elements in an open neighborhood of $e \in G$ (contained in $\exp(\mathfrak{g}) \subseteq G$). Since G is connected, every element $h \in G$ is of the form $h = h_1h_2\cdots h_n$ for h_i in the neighborhood, and since g commutes with h_i individually, it commutes with h, so $g \in \mathrm{Z}(G)$.

(2) If G is connected, $\mathrm{Z}(G)$ is a closed subgroup and

$$\mathrm{Lie}(\mathrm{Z}(G)) = \mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} \colon \forall Y \in \mathfrak{g}, [X, Y] = 0\}.$$

Solution. Note that $\mathrm{Z}(G) = \{g \in G \colon \forall h \in G, ghg^{-1}h^{-1} = e\}$, so we can write

$$\mathrm{Z}(G) = \bigcap_{h \in G} f_h^{-1}(e) \quad \text{for} \quad f_h(g) = ghg^{-1}h^{-1}$$

as a closed subgroup. By Corollary 3.97 we have

$$\mathrm{Lie}(\mathrm{Z}(G)) = \mathrm{Lie}(\mathrm{Ker}(\mathrm{Ad})) = \mathrm{Ker}(\mathrm{D}\,\mathrm{Ad}) = \mathrm{Ker}(\mathrm{ad}) = \mathfrak{z}(\mathfrak{g})$$

since $\mathrm{ad}_X(Y) = [X, Y]$.

Exercise 3 (Quotients of Lie groups)**.** Let G be a Lie group and let $K \leq G$ be a closed normal subgroup. Show that G/K can be equipped with a Lie group structure such that the quotient map $\pi \colon G \to G/K$ is a surjective Lie group homomorphism with kernel K.

Solution. From the lecture we know that there exists a suitable neighborhood $U \subset \mathfrak{g}$ of the origin such that $\exp|_U : U \to \exp(U)$ is a diffeomorphism. Denote by $\mathfrak{k} = \mathrm{Lie}(K)$ the Lie algebra associated to K. Choose any complement $\mathfrak{l}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$ as vector spaces. Define

$$V := U \cap \mathfrak{l}.$$

Since $V \cap \mathfrak{k} = \{0\}$ it is immediate to verify that $\pi \circ \exp|_V : V \to G/K$ is a homeomorphism onto the image. This gives us a local chart around the point $K \in G/K$. We can get an atlas by suitably translating this chart by the natural action of G on G/K. This gives us back an atlas such that each change of coordinate charts is smooth (since the multiplication in G is smooth).

Note that multiplication and inversion are defined on G/K by passing to the quotient, i.e. the following diagrams commute:

$$\begin{array}{ccc} G \times G & \xrightarrow{\;m\;} & G \\ \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\ G/K \times G/K & \dashrightarrow & G/K \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\;i\;} & G \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G/K & \dashrightarrow & G/K \end{array}$$

By definition, the quotient map $\pi \colon G \to G/K$ is a smooth submersion with respect to this smooth structure. Thus, it follows from the constant rank theorem that multiplication and inversion are smooth, and G/K is a Lie group. Moreover, it is clear from the construction that K is the kernel of π.

For more details see Theorem 21.26 in *John M. Lee, "Intorduction to Smooth Manifolds", Springer (2013)*

Exercise 4 (Connectedness from quotients)**.** Let G be a topological group and $H < G$ a closed subgroup. Show that if H and G/H are connected, then so is G.

Solution. We suppose that H and G/H are connected and that $G = A \cup B$ for disjoint, non-empty open sets A and B in G. Assume without loss of generality that $e \in A$. Since H is connected, all of its left cosets $gH = L_g(H)$ are. Thus since each coset meets either A or B it must be contained entirely in one of the two. Consequently, A and B are union of left cosets of H. If now $p\colon G \to G/H$ denotes the projection map on left cosets, it follows that both $p(A)$ and $p(B)$ are non-empty disjoint. Since p is open, $p(A)$ and $p(B)$ are open non-empty disjoint whose union is G/H, which contradicts the connectedness of G/H.

Exercise 5 (Examples of solvable and nilpotent groups)**.** Compute the derived series and the central series of the Lie groups

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b \neq 0 \right\}, \qquad H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathrm{SL}(2,\mathbb{R})$$

to decide whether they are are solvable and/or nilpotent. Find all weights for the inclusion-representations $\rho_G\colon G \to \mathrm{GL}(2,\mathbb{C})$, $\rho_H\colon H \to \mathrm{GL}(3,\mathbb{C})$ and $\rho_{\mathrm{SL}(2,\mathbb{R})}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(2,\mathbb{C})$.

Solution. The derived series of a group G is defined by

$$G^{(i)} = [G^{(i-1)}, G^{(i-1)}] := \langle ghg^{-1}h^{-1} \colon g, h \in G^{(i-1)} \rangle,$$

while the central series is defined by $C^i(G) = [G, C^{i-1}]$. For

$$g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, h = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \in G$$

we have

$$ghg^{-1}h^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \begin{pmatrix} a^{-1} & -\frac{b}{ac} \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} d^{-1} & -\frac{e}{df} \\ 0 & f^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{fc-bd-ae-bf-ec}{cf} \\ 0 & 1 \end{pmatrix},$$

so

$$G^{(0)} = G, \qquad G^{(1)} = \left\{ \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad G^{(2)} = \{\mathrm{Id}\},$$

whence G is solvable. A similar calculation shows that H is solvable with derived series

$$H^{(0)} = H, \qquad H^{(1)} = \left\{ \begin{pmatrix} 1 & 0 & \star \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad H^{(2)} = \{\mathrm{Id}\},$$

For $\mathrm{SL}(2,\mathbb{R})$ it is not very practical to multiply out a general element $ghg^{-1}h^{-1}$, instead we note that

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2-1 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ a^2-1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}$$

These three elements together with their inverses generate a neighborhood of Id and since $\mathrm{SL}(2,\mathbb{R})$ is connected, they generate the whole group, so $\mathrm{SL}(2,\mathbb{R})^{(i)} = \mathrm{SL}(2,\mathbb{R})$ and thus $\mathrm{SL}(2,\mathbb{R})$ is not solvable.

For the central series we obtain

$$C^0(G) = G, \qquad C^1(G) = G^{(1)} = \left\{ \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad C^2(G) = [G, C^1(G)] = C^1(G) \dots$$
$$C^0(H) = H, \qquad C^1(H) = H^{(0)} \quad \text{and} \quad C^2(H) = \{\mathrm{Id}\}$$
$$C^i(\mathrm{SL}(2,\mathbb{R})) = \mathrm{SL}(2,\mathbb{R})$$

so H is nilpotent, while G and $\mathrm{SL}(2,\mathbb{R})$ are not.

A *weight* of a representation $G \to \mathrm{GL}(V)$ for a complex vector space V is a group homorphism $\chi\colon G \to \mathbb{C}^\star$ such that

$$V_\chi := \{X \in V \colon \rho(g)(X) = \chi(g) \cdot X \text{ for all } g \in G\} \neq \{0\}.$$

For $G \subseteq \mathrm{GL}(2,\mathbb{C})$, if

$$\rho(g) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cv \end{pmatrix} = \chi(g) \cdot \begin{pmatrix} u \\ v \end{pmatrix},$$

then $v = 0$ and $\chi(g) = a$, so the only weight is

$$\chi\colon \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a \quad \text{with} \quad V_\chi = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \neq \{0\}.$$

For H we have

$$\rho(g) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + av + bw \\ v + cw \\ w \end{pmatrix} = \chi(g) \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

then $w = 0$ and $v = 0$, so $\chi(g) = 1$, so the only weight is the constant one weight χ_1 with weight space

$$V_{\chi_1} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \neq \{0\}.$$

Finally for $\mathrm{SL}(2,\mathbb{R})$ there are no weights at all.