Solutions to exercise Sheet 7

Exercise 1 (Killing form of $\mathfrak{sl}(2,\mathbb{R})$)**.** Choose a basis of $\mathfrak{sl}(2,\mathbb{R})$ to compute the Killing form $K_{\mathfrak{sl}(2,\mathbb{R})}(X,Y) = 4\mathrm{tr}(XY)$.

Solution. Recall that $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{R}) = \left\{X \in \mathbb{R}^{2\times 2} \colon \mathrm{tr}X = 0\right\}$. We choose a basis

$$X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which allows us to identify $\mathfrak{gl}(\mathfrak{g}) \cong \mathfrak{gl}(3,\mathbb{R})$. We want to compute the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. We compute

$$\left[\begin{pmatrix} b & a \\ c & -b \end{pmatrix}, X_1\right] = \begin{pmatrix} -c & 2b \\ 0 & c \end{pmatrix},$$
$$\left[\begin{pmatrix} b & a \\ c & -b \end{pmatrix}, X_2\right] = \begin{pmatrix} 0 & -2a \\ 2c & 0 \end{pmatrix},$$
$$\left[\begin{pmatrix} b & a \\ c & -b \end{pmatrix}, X_3\right] = \begin{pmatrix} a & 0 \\ -2b & -a \end{pmatrix},$$

so

$$\mathrm{ad}\begin{pmatrix} b & a \\ c & -b \end{pmatrix} = \begin{pmatrix} 2b & -2a & 0 \\ -c & 0 & a \\ 0 & 2c & -2b \end{pmatrix} \in \mathfrak{gl}(\mathfrak{g}).$$

The Killing form is defined as $K_{\mathfrak{g}}(X,Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ and on the basis we calculate

$$K_{\mathfrak{g}}\left(\begin{pmatrix} b & a \\ c & -b \end{pmatrix}, X_1\right) = \mathrm{tr}\begin{pmatrix} 2b & -2a & 0 \\ -c & 0 & a \\ 0 & 2c & -2b \end{pmatrix}\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{tr}\begin{pmatrix} 0 & -4b & -2a \\ 0 & 2c & 0 \\ 0 & 0 & 2c \end{pmatrix} = 4c$$
$$K_{\mathfrak{g}}\left(\begin{pmatrix} b & a \\ c & -b \end{pmatrix}, X_2\right) = \mathrm{tr}\begin{pmatrix} 2b & -2a & 0 \\ -c & 0 & a \\ 0 & 2c & -2b \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \mathrm{tr}\begin{pmatrix} 4b & 0 & 0 \\ -2c & 0 & -2a \\ 0 & 0 & 4b \end{pmatrix} = 8b$$
$$K_{\mathfrak{g}}\left(\begin{pmatrix} b & a \\ c & -b \end{pmatrix}, X_3\right) = \mathrm{tr}\begin{pmatrix} 2b & -2a & 0 \\ -c & 0 & a \\ 0 & 2c & -2b \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \mathrm{tr}\begin{pmatrix} 2a & 0 & 0 \\ 0 & 2a & 0 \\ -2c & -4b & 0 \end{pmatrix} = 4a,$$

so writing the Killing form as a bilinear form in the basis we obtain

$$K_{\mathfrak{g}}(X,Y) = X^\top \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix} Y$$

for $X, Y \in \mathfrak{g} \cong \mathbb{R}^3$. Actually a calculation then shows that also $K_{\mathfrak{g}}(X,Y) = 4\mathrm{tr}(XY)$.

Exercise 2 (The Killing form is invariant under the adjoint action)**.** Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Prove that for all $X, Y \in \mathfrak{g}$ and $g \in G$

$$K_{\mathfrak{g}}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = K_{\mathfrak{g}}(X, Y).$$

Hint: Prove that $\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g)\,\mathrm{ad}(X)\,\mathrm{Ad}(g)^{-1}$.

Solution. Let $X, Y, Z \in \mathfrak{g}$ and $g \in G$. Since $\mathrm{Ad}(g)$ is the derivative of the Lie group automorphism c_g, Ad is a Lie algebra automorphism, so $\mathrm{Ad}(g)[X, Y] = [\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y]$. Then $\mathrm{ad}(\mathrm{Ad}(g)X)(Z) = [\mathrm{Ad}(g)X, Z] = \mathrm{Ad}(g)[X, \mathrm{Ad}(g)^{-1}Z] = \mathrm{Ad}(g)\,\mathrm{ad}(Y)\,\mathrm{Ad}(g)^{-1}$ proving the hint. Then

$$\begin{aligned} K_{\mathfrak{g}}(\mathrm{ad}(\mathrm{Ad}(g)X) \circ \mathrm{ad}(\mathrm{Ad}(g)Y)) &= \mathrm{tr}\left(\mathrm{Ad}(g)\,\mathrm{ad}(X)\,\mathrm{Ad}(g)^{-1}\,\mathrm{Ad}(g\,\mathrm{ad}(Y)\,\mathrm{Ad}(g)^{-1})\right) \\ &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = K_{\mathfrak{g}}(X, Y) \end{aligned}$$

since the trace is invariant under conjugation.

Exercise 3 (Solvable Lie group without injective finite-dimensional representation)**.** Consider the three-dimensional Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Note that the center of H is

$$\mathrm{Z}(H) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$

Let $D < \mathrm{Z}(H)$ be the following discrete subgroup

$$D := \mathrm{SL}_3(\mathbb{Z}) \cap \mathrm{Z}(H) = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Check that $G := H/D$ is a connected, solvable Lie group and show that G does not admit a smooth, injective homomorphism into $\mathrm{GL}(V)$ for any finite-dimensional $\mathbb{C}$-vector space V.

Hint: Observe that $\mathrm{Z}(H)/D \cong S^1$ and consider its image under a potential representation and show that its image can be conjugated into any small neighborhood of $\mathrm{Id} \in \mathrm{GL}(n, \mathbb{C})$. Then use the no-small-subgroups property.

Solution. Since H is connected, so is G, and since both H and D are solvable, so is G. Assume $\pi \colon G \to \mathrm{GL}(V)$ is a smooth homomorphism for a finite-dimensional $\mathbb{C}$-vector space V. We will show that $\pi(Z(H)/D) = \mathrm{Id}$, that is $Z(H)/D < \ker \pi$, so that π cannot be injective.

Let us observe first of all that, since D is discrete, then

$$\mathrm{Lie}(H/D) = \mathrm{Lie}(H) = \mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \subset \mathfrak{gl}(3, \mathbb{R})$$

and it is moreover solvable. Furthermore

$$\mathrm{Lie}(Z(H)/D) = \mathrm{Lie}(Z(H)) = \left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = [\mathfrak{h}, \mathfrak{h}] \,.$$

By Lie's theorem, if $\rho := d_e\pi$, the image $\rho(\mathfrak{h})$ is upper triangular, so that $[\rho(\mathfrak{h}), \rho(\mathfrak{h})]$ is strictly upper triangular. Thus

$$\rho(\mathrm{Lie}(Z(H)/D)) = \rho([\mathfrak{h}, \mathfrak{h}]) = [\rho(\mathfrak{h}), \rho(\mathfrak{h})] \subset \left\{ \begin{pmatrix} 0 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 0 \end{pmatrix} \right\},$$

from which it follows that

$$\pi(Z(H)/D) < \left\{ \begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \right\} =: L \,.$$

Observe that since $Z(H)/D \simeq S^1$, then $\pi(Z(H)/D) =: K$ is a compact subgroup of L. We will show now that L cannot have non-trivial compact subgroups, which forces $K = \mathrm{Id}$. In order to show this, we will show that any compact subgroup can be conjugated into any small neighborhood of $\mathrm{Id} \in \mathrm{GL}(n, \mathbb{C})$, thus contradicting that L is a Lie group.

To this purpose, let $g = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathrm{GL}(n, \mathbb{C})$ a diagonal matrix with entries $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$. Then, if $i < j$,

$$\begin{aligned} \left(c_g \begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \right)_{ij} &= \left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix} \right) \\ &= \frac{\lambda_i}{\lambda_j} \begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix}_{ij}, \end{aligned}$$

so that

$$\left(c_g^n \begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \right)_{ij} = \left(\frac{\lambda_i}{\lambda_j} \right)^n \begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix}_{ij} \tag{1}$$

If $\begin{pmatrix} 1 & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \in K$ its entries are bounded and, since $\lambda_i/\lambda_j < 1$, the right hand side of (1) converges to Id and is hence eventually contained in any neighborhood of Id, no matter how small.

Exercise 4 (Complex Lie algebras)**.** Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ the complexification of $\mathfrak{g}$ as a vector space.

(1) Show that the bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ extends uniquely to a $\mathbb{C}$-bilinear map $[\cdot,\cdot]_\mathbb{C}\colon \mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ turning $\mathfrak{g}_\mathbb{C}$ into a complex Lie algebra.

Solution. It is clear that $\mathfrak{g}_\mathbb{C}$ is a $\mathbb{C}$-vector space. The universal property of the tensor product states that for every bilinear map $h\colon \mathfrak{g} \times \mathbb{C} \to V$ for $\mathbb{R}$-vectorspace V, there exists a unique linear map $\bar{h}\colon \mathfrak{g}_\mathbb{C} \to Z$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{g} \times \mathbb{C} & \xrightarrow{h} & V \\ \otimes\big\downarrow & \nearrow_{\exists! \bar{h}} & \\ \mathfrak{g}_\mathbb{C} & & \end{array}$$

Solution. For any $X \in \mathfrak{g}$ and $z \in \mathbb{C}$ we have a bilinear map $h\colon \mathfrak{g} \times \mathbb{C} \to \mathfrak{g}_\mathbb{C}$ defined by $h_{X,z}(Y, w) := [X, Y] \otimes zw$. By the universal property there is a unique map $\bar{h}_{X,z}\colon \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$. Now consider the map $h'\colon \mathfrak{g} \times \mathbb{C} \to \mathrm{End}(\mathfrak{g}_\mathbb{C})$ defined by $h'(X, z) := \bar{h}_{X,z}$ and notice that it is also bilinear and thus by the universal property has a unique extension $\bar{h}'\colon \mathfrak{g}_\mathbb{C} \to \mathrm{End}(\mathfrak{g}_\mathbb{C})$. We define the bracket on generators as

$$\begin{aligned} &[\cdot,\cdot]_\mathbb{C}\colon \mathfrak{g}_\mathbb{C} \times \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C} \\ &(X \otimes z, Y \otimes w) \mapsto \bar{h}'(X \otimes z)(Y \otimes w) = [X, Y] \otimes zw \end{aligned}$$

It is clear that the brackets are bilinear. Next, $[X \otimes z, X \otimes z]_\mathbb{C} = [X, X] \otimes z^2 = 0 \otimes z^2 = 0$ since $[X, X] = 0$. And finally, the Jacobi equality

$$\begin{aligned} &[X \otimes z, [Y \otimes w, Z \otimes v]_\mathbb{C}]_\mathbb{C} + [Y \otimes w, [Z \otimes v, X \otimes z]_\mathbb{C}]_\mathbb{C} + [Z \otimes v, [X \otimes z, Y \otimes w]_\mathbb{C}]_\mathbb{C} \\ =&[X, [Y, Z]] \otimes zwv + [Y, [Z, X]] \otimes wvz + [Z, [X, Y]] \otimes vzw \\ =&([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]) \otimes zwv = 0 \otimes zvw = 0 \end{aligned}$$

holds on generators, so it holds always. This shows that $\mathfrak{g}_\mathbb{C}$ is a complex Lie algebra.

(2) Show that the canonical injection $\mathfrak{g} \to \mathfrak{g}_\mathbb{C}$, $X \mapsto X \otimes 1$ is a homomorphism of real Lie algebras and, if we identify $\mathfrak{g}$ with its image in $\mathfrak{g}_\mathbb{C}$, we have that

$$\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g}.$$

Express the bracket of $\mathfrak{g}_\mathbb{C}$ in this decomposition.

Solution. Every complex Lie algebra can be viewed as a real Lie algebra by considering the $\mathbb{C}$-vectorspace as a $\mathbb{R}$-vectorspace and taking the same brackets. The map $f\colon \mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}, X \mapsto X \otimes 1$ is linear

$$f(\lambda X + Y) = (\lambda X + Y) \otimes 1 = \lambda(X \otimes 1) + Y \otimes 1 = \lambda f(X) + f(Y)$$

and preserves the brackets

$$f([X,Y]) = [X,Y] \otimes 1 = [X \otimes 1, Y \otimes 1]_{\mathbb{C}} = [f(X), f(Y)]_{\mathbb{C}}$$

and is hence a homomorphism of real Lie algebras. The map f is clearly an injection. Let

$$h\colon \mathfrak{g} + i\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$$

defined on generators as $h(X + iY) = X \otimes 1 + i(Y \otimes 1) = X \otimes 1 + Y \otimes i$. Clearly h is linear and $h^{-1}\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g} + i\mathfrak{g}$ defined on generators as $h^{-1}(X \otimes z) := X \otimes x + i(X \otimes y)$ for $z = x + iy$ is a linear map and they are inverses of each other. If now $X + iY, X' + iY' \in \mathfrak{g} + i\mathfrak{g}$, then

$$\begin{aligned}
[X + iY, X' + iY']_{\mathbb{C}} &= [X \otimes 1 + Y \otimes i, X' \otimes 1 + Y' \otimes i]_{\mathbb{C}} \\
&= [X \otimes 1, X' \otimes 1]_{\mathbb{C}} + [X \otimes 1, Y' \otimes i]_{\mathbb{C}} + [Y \otimes 1, X' \otimes 1]_{\mathbb{C}} + [Y \otimes 1, Y' \otimes i]_{\mathbb{C}} \\
&= [X, X'] \otimes 1 + [X, Y'] \otimes i + [Y, X'] \otimes i + [Y, Y'] \otimes i^2 \\
&= ([X, X'] - [Y, Y']) \otimes 1 + ([X, Y'] + [Y, X']) \otimes i \\
&= ([X, X'] - [Y, Y']) + i([X, Y'] + [Y, X']).
\end{aligned}$$

(3) Show that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_{\mathbb{C}}$ is solvable,

Solution. If $\mathfrak{h} \lhd \mathfrak{g}$ is an ideal, then $\mathfrak{h}_{\mathbb{C}} =: \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C} \lhd \mathfrak{g}_{\mathbb{C}}$ is an ideal. So any series of ideals

$$\mathfrak{g} \rhd \mathfrak{g}_1 \rhd \mathfrak{g}_2 \rhd \ldots \rhd \{0\}$$

corresponds to a series of ideals

$$\mathfrak{g}_{\mathbb{C}} \rhd (\mathfrak{g}_1)_{\mathbb{C}} \rhd (\mathfrak{g}_2)_{\mathbb{C}} \rhd \ldots \rhd \{0\}.$$

If $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian, then $[X + \mathfrak{g}_{i+1}, Y + \mathfrak{g}_{i+1}] = [X,Y] + \mathfrak{g}_{i+1} = 0 \in \mathfrak{g}_i/\mathfrak{g}_{i+1}$ for all $X, Y \in \mathfrak{g}_i$. For $X + iY + (\mathfrak{g}_{i+1})_{\mathbb{C}}, X' + iY' + (\mathfrak{g}_{i+1})_{\mathbb{C}} \in (\mathfrak{g}_i)_{\mathbb{C}}/(\mathfrak{g}_{i+1})_{\mathbb{C}}$ we then have

$$\begin{aligned}
[X + iY + (\mathfrak{g}_{i+1})_{\mathbb{C}}, X' + iY' + (\mathfrak{g}_{i+1})_{\mathbb{C}}]_{\mathbb{C}} &= [X + iY, X' + iY']_{\mathbb{C}} + (\mathfrak{g}_{i+1})_{\mathbb{C}} \\
&= ([X, X'] - [Y, Y']) + i([X, Y'] + [Y, X']) + (\mathfrak{g}_{i+1})_{\mathbb{C}} \\
&= 0 \in (\mathfrak{g}_i)_{\mathbb{C}},
\end{aligned}$$

so if $\mathfrak{g}$ is solvable, then $\mathfrak{g}_{\mathbb{C}}$ is solvable.

If we start with an ideal $\mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$ on the other hand, then $\mathfrak{h} := \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{g}$ is an ideal in $\mathfrak{g}$. In fact, since $\mathfrak{h}_{\mathbb{C}}$ is invariant under multiplication of $\mathbb{C}$, we have that $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} + i\mathfrak{h}$ as before. Then starting from a series of ideals

$$\mathfrak{g}_{\mathbb{C}} \rhd (\mathfrak{g}_1)_{\mathbb{C}} \rhd (\mathfrak{g}_2)_{\mathbb{C}} \rhd \ldots \rhd \{0\}$$

we obtain a series of ideals

$$\mathfrak{g} \rhd \mathfrak{g}_1 \rhd \mathfrak{g}_2 \rhd \ldots \rhd \{0\}.$$

If $(\mathfrak{g}_i)_\mathbb{C}/(\mathfrak{g}_{i+1})_\mathbb{C}$ is abelian, it means that for every $X+iY, X'+iY' \in (\mathfrak{g}_i)_\mathbb{C}$ we have $[X+iY, X'+iY']_\mathbb{C} + \mathfrak{g}_{i+1} = 0$. For $X, Y \in \mathfrak{g}_i$ we then have $[X + \mathfrak{g}_{i+1}, Y + \mathfrak{g}_{i+1}] \in (\mathfrak{g}_{i+1})_\mathbb{C} = \mathfrak{g}_{i+1} + i\mathfrak{g}_{i+1}$. But since $[\mathfrak{g}, \mathfrak{g}] \cap i\mathfrak{g}_{i+1} = 0$, we have that $[X + \mathfrak{g}_{i+1}, Y + \mathfrak{g}_{i+1}] = 0 \in \mathfrak{g}_i/\mathfrak{g}_{i+1}$. So if $\mathfrak{g}_\mathbb{C}$ is solvable, then so is $\mathfrak{g}$.

(4) Show that $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}_\mathbb{C}$ is nilpotent.

Solution. If $\mathfrak{g}$ is nilpotent, then in the notation of (3), $[\mathfrak{g}, \mathfrak{g}_{i+1}] \subseteq \mathfrak{g}_i$. So if $X + iY \in \mathfrak{g}_\mathbb{C}$ and $X' + iY' \in (\mathfrak{g}_{i+1})_\mathbb{C}$, then

$$[X + iY, X' + iY']_\mathbb{C} = ([X, X'] - [Y, Y']) + i([X, Y'] + [Y, X']) \in \mathfrak{g}_i + i\mathfrak{g}_i = (\mathfrak{g}_i)_\mathbb{C}$$

as required.

If however $\mathfrak{g}_\mathbb{C}$ is nilpotent, then $[\mathfrak{g}_\mathbb{C}, (\mathfrak{g}_{i+1})_\mathbb{C}] \subseteq (\mathfrak{g}_i)_\mathbb{C}$. Since $\mathfrak{g} \subseteq \mathfrak{g}_\mathbb{C}$ and $\mathfrak{g}_{i+1} \subseteq (\mathfrak{g}_{i+1})_\mathbb{C}$, we have that $[\mathfrak{g}, \mathfrak{g}_{i+1}] \subseteq (\mathfrak{g}_i)_\mathbb{C} \cap \mathfrak{g} = \mathfrak{g}_i$, so $\mathfrak{g}$ is nilpotent.