Introduction to Lie groups, diary of lectures

May 17, 2024

Important note: unless otherwise stated, the proofs of Theorems, Propositions, Lemmas, and Corollaries are part of the program (if they were discussed in class).

- Lecture 1: 21/02
  - Brief historical introduction to Lie groups; Hilbert's fifth problem; Hilbert-Smith conjecture;
  - Structure of the course;
  - Definition of Topological group and elementary consequences;
  - Examples of topological groups; $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{Iso}(X)$, $\mathrm{Diff}(M)$.
  - Matrix groups: $\mathrm{O}(n, \mathbb{R})$, A, N, $\mathrm{O}(p, q)$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{U}(n)$, $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{SO}(n)$, $\mathrm{SO}(p, q)$, $\mathrm{SU}(n)$.
- Lecture 2: 22/02
  - Compactness and local compactness;
  - Examples of compact and locally compact topological groups;
  - $\mathrm{O}(n)$ is compact, $\mathrm{O}(p, q)$ for $p, q \geq 1$ is not;
  - Statement of Proposition 2.31 about general properties of topological groups.
  - Statement and proof of Lemma 2.32 (existence of "nice" neighborhoods of the identity).
  - Proof of Prop. 2.31 (up to item 4).
- Lecture 3: 28/02
  - Proof of item 5 of Prop. 2.31;
  - The set of connected components and its group structure; examples.
  - Existence of the universal covering for locally path-connected semi locally simply connected top. spaces (statement only);
  - the fundamental group of a locally path connected and semi locally simply connected top group is abelian.
  - Local homomorphisms; definition and examples.
  - Theorem 2.42: extension of local homomorphisms from simply connected top. groups. Statement and proof up to the end of Step 1.
- Lecture 4: 29/02 (Exercise class)
  - Intuition for the compact-open topology.

- On metric spaces the compact-open topology coincides with uniform convergence on compact sets.
- An example of a topological space X, such that $\mathrm{Homeo}(X)$ is not a topological group. (using Cantor set)
- If X compact, then $\mathrm{Homeo}(X)$ is a topological group.

- Lecture 5: 06/03
  - Completion of the proof of Prop. 2.31
  - Statement of Corollary 2.38
  - Introduction to Haar measures, statement of the existence and uniqueness theorem for the Lebesgue measure
  - left group actions, continuous actions
  - endomorphism of $C_c(X)$ induced by G-action;
  - Statement of Riesz representation theorem.
  - Induced action on linear functionals
  - Definition of left Haar functional and measure
  - Statement of Haar's theorem (Thm. 2.42).
  - Relationship between left and right Haar measure, Lemma 2.44 and Corollary 2.45.
  - σ-finiteness of Haar measure on locally compact Hausdorff connected groups (Lemma 2.46).
- Lecture 6: 07/03
  - Statement and proof of Lemma 2.47 (general properties of Haar measures)
  - Proof of uniqueness part in Haar's theorem
  - Examples of Haar measures: Lebesgue, Haar measure on $\mathbb{R}_{>0}$, counting measure on discrete groups.
  - Mentioned existence of left Haar measures that are not right Haar measures. Left Haar=right Haar does not imply abelian.
  - The group $\mathrm{Aut}(G)$; left action of $\mathrm{Aut}(G)$ on $\mathrm{C_c(G)}$
  - Statement and proof of Lemma 2.53; definition of $\mathrm{mod_G}$.
  - Statement and proof of Lemma 2.54: $\mathrm{mod_G} : \mathrm{Aut(G)} \to \mathbb{R}_{>0}$ is a homomorphism.
  - Inner automorphisms, construction of modular function $\Delta_G : G \to \mathbb{R}_{>0}$.
  - Modular function measures how non-right invariant a left invariant measure is.
  - Statement of Prop. 2.55
  - Definition of left/right uniformly continuous function (Def. 2.57)
  - Statement and proof of Lemma 2.57, any $f \in \mathrm{C}_c(G)$ is both left and right unif. continuous.
- Lecture 7: 13/03
  - Corollary 2.58, statement and proof
  - Proof of Proposition 2.55

  - Definition of unimodular group and examples
  - Corollary 2.61, statement and proof
  - Prop. 2.62, statement and proof
  - Further examples of unimodular groups (abelian and compact, Ex. 2.63)
  - Introduction to homogeneous spaces.
  - Topological properties, Prop. 2.64, statement and proof.
  - Examples of homogeneous spaces, Ex. 2.65: S^n and GO_k.

- Lecture 8: 14/03 (Exercise class)
  - Sheet 1, Exercise 4 about $\mathrm{GL}(n, \mathbb{R})$. One takaway is the method of showing that something is closed by viewing it as an intersection of preimages of a closed set.
  - Sheet 1, Exercise 5 about $\mathrm{Iso}(X)$ being a compact group, when X is a compact metric space.
  - Sheet 1, Exercise 7: topological coverings of topological groups are topological groups.
  - Fundamental group of a topological group is abelian, so $\mathbb{R}^2 \setminus \{p_1, p_2\}$ does not admit a topological group structure.
  - Examples of obtaining new groups from given ones. a) Quotienting by a normal subgroup, b) taking covers, in particular universal covers.
  - Excursion about the appearance of the universal covering of $\mathrm{SL}(2, \mathbb{R})$ in the Thurston geometrization conjecture as one of the model geometries.

- Lecture 9: 20/03
  - Remark 2.71: the stabilizer might depend on the point; if the group acts transitively then stabilizers of different points are conjugated; the stabilizer of a point might be not normal; G/G_x might not admit any topological group structure; in general $G/G_x \times G_x$ is not homeomorphic to G.
  - Proposition 2.73: compact subgroups of $\mathrm{GL}(n, \mathbb{R})$ are conjugated to subgroups of $\mathrm{O}(n, \mathbb{R})$.
  - Definition 2.66 (Lattice).
  - Discussion about the definition of lattice, see Exercise 2.72.
  - Theorem 2.68, statement only.
  - Discussion on the well-posedness of Weil's formula.
  - Proposition 2.70: group with lattice must be unimodular.

- Lecture 10: 21/03
  - More details about the proof of Prop 2.70
  - Introduction to the chapter about Lie groups;
  - Definition 3.1 (Lie group)
  - Definition 3.2 (Top. manifold), Definition 3.3 (smooth structure)
  - Remark 3.4 (Top manifolds are paracompact and have countably many connected components)
  - Definition of smooth map between smooth manifolds

- Examples 3.5, 3.6. 3.7 of Lie groups
  - Example 3.8 (Homeo(X) is not Lie in general) and Example 3.9 (Iso(X) might or might not be Lie)
  - Definition 3.10 (Regular submanifold).
  - Theorem 3.11 (statement only). Exercise 3.12 (statement).
  - Theorem 3.13 (statement only)
  - First part of Example 3.14: $\mathrm{SL}(\mathrm{n}, \mathbb{R})$ is a Lie group

- Lecture 11: 27/03
  - Second part of Example 3.14: $\mathrm{O}(\mathrm{n}, \mathbb{R})$ is a Lie group
  - Definition 3.16 (tangent vectors)
  - Definition 3.18 (vector field and smooth vector field)
  - Discussion about the expression of vector fields in local coordinates
  - Definition 3.19 (derivation of an algebra)
  - Proposition 3.20 (Isomorphism between $\mathrm{Vect}^\infty(\mathrm{M})$ and $\mathrm{Der}(\mathrm{C}^\infty(\mathrm{M}))$)
  - Remark 3.21: representing germ with global smooth function and vice versa
  - Remark: composition of derivations is not a derivation in general
  - Lemma 3.22
  - Definition 3.23: bracket of vector fields
  - Definition 3.24: bracket of endomorphisms.

- Lecture 12: 28/03 (Exercise class)
  - Sheet 2, Exercise 2c) Haar measure on $\mathrm{GL}(n, \mathbb{R})$
  - Sheet 2, Exercise 2d) Haar measure on $\mathrm{SL}(n, \mathbb{R})$
  - Recap lattices. Fundamental domains. Examples
  - The modular group $\mathrm{SL}(2, \mathbb{Z})$ as a lattice that is not cocompact, via its action on the hyperbolic plane $\mathbb{H} = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$.

- Lecture 13: 10/04
  - Properties of the bracket of endomorphisms
  - Remark 3.25: Jacobi identity is a substitute of associativity
  - Definition 3.26 (Lie algebra)
  - Example 3.27 (examples of Lie algebras)
  - Definition 3.28 (Lie algebra homomorphism)
  - Recap about definition of differential of a smooth map
  - Discussion about how to use a smooth map to “push-forward” vector fields
  - Definition 3.28 (φ-related vector fields)
  - Lemma 3.30, algebraic characterization of φ-related vector fields (no proof);
  - Proposition 3.31: brackets of φ-related vector fields are φ-related
  - Definition of push-forward of vector field with a diffeomorphism

  - Corollary 3.32: diffeomorphism induces Lie algebra isomorphism between spaces of vector fields
  - Recap: useful identifications for tangent spaces of vector spaces and differentials of linear maps
  - Definition 3.33: smooth action, induced left translation diffeomorphisms
  - Definition 3.34: G-invariant vector fields
  - Definition 3.35: Lie subalgebra, left invariant vector fields are a Lie subalgebra
  - Lemma 3.36
  - Definition 3.37: Lie algebra of a Lie group
  - Proposition 3.38, about the Lie algebra of $\mathrm{GL}(\mathrm{n}, \mathbb{R})$

- Lecture 14: 11/04
  - Proof of Proposition 3.38
  - Proposition 3.39: differential of Lie groups homomorphism is Lie algebra homomorphism
  - Corollary 3.40
  - Example 3.41 of Lie algebras of matrix groups
  - Example 3.43: Lie algebra of product of Lie groups
  - Broad discussion about the functor from Lie groups to Lie algebras and what comes next.
  - Examples of non-isomorphic Lie groups with isomorphic Lie algebras
  - Discussion about Cartan's theorem
  - The matrix exponential map, Proposition 3.44.
  - Sketch of proof of 1) and 2). Proof of 3).

- Lecture 15: 17/04
  - Proof of Proposition 3.44 part 4.
  - Definition 3.45, integral curve.
  - Theorem 3.46, existence and uniqueness of maximal integral curves of smooth vector fields (no proof)
  - Definition 3.47: complete vector field
  - Proposition 3.48: group law for the flow map
  - Discussion about rate of change of vector fields on smooth manifolds
  - Definition 3.49: Lie derivative
  - Theorem 3.50: Lie derivative = Lie bracket (no proof)
  - Proposition 3.51: flows commute iff vector fields commute (no proof)
  - Proposition 3.52: flows of left invariant vector fields on Lie groups
  - Definition 3.53: one parameter group
  - Corollary 3.54: one parameter groups are all obtained via Prop. 3.52
  - Definition 3.56: exponential map
  - Corollary 3.57: statement only.

- Lecture 16: 18/04 (exercise Class)
  - Sheet 3, Ex 1 about lattices.
  - Sheet 3, Ex 2, Lie subgroups that are regular submanifolds are closed.
  - Sheet 3, Ex 3 about tangent space being isomorphic to $\mathbb{R}^n$.
  - various notions of tangent space, including one about paths through p.
  - S^2 is not a Lie group.
- Lecture 17: 24/04
  - Lemma 3.58: differential of multiplication.
  - Proof of Corollary 3.57: properties of the exponential.
  - Proposition 3.59: exponential and Lie group homomorphisms
  - Corollary 3.61: exponential as a preferred chart near to e.
  - Discussion about surjectivity of $\exp_{\mathrm{G}}$: connectedness is necessary, Cartan's theorem 3.62 (statement only), example of connected Lie group with non-surjective exponential (Example 3.64).
  - Definition 3.65: abelian Lie algebra
  - Proposition 3.66: structure of abelian Lie groups
  - Introduction to the correspondence between Lie subalgebras and Lie subgroups
  - Definition 3.72: immersion, immersed vs embedded submanifolds
  - Example 3.73: immersions and immersed sumanifolds
  - Example 3.75: some Lie subgroups of T^2
  - Definition 3.76: Lie subgroup.
- Lecture 18: 25/04
  - Theorem 3.77: Lie subalgebras-Lie subgroups correspondence
  - Remark 3.78: about the uniqueness part of Thm 3.77
  - Introductory discussion about distributions and integral submanifolds
  - Definition 3.80: distribution, involutive distribution, integral submanifold, completely integrable distribution
  - Example 3.81: distributions and integral submanifolds
  - Theorem 3.82: Frobenius
  - Remark 3.83: the 1 dimensional case
  - Definition 3.84: maximal integral submanifold
  - Theorem 3.85: esistence and uniqueness of maximal integral submanifolds through each point for involutive distributions
  - Sketch of proof of Theorem 3.77
  - Theorem 3.87
  - Example 3.88: Lie algebra homomorphism not induced by Lie group homomorphism
  - Theorem 3.89: Li algebra homomorphism is induced by local Lie group homomorphism

- Lemma 3.90 (no proof)
- Proof of Theorem 3.89
- Theorem 3.91: Ado's theorem
- Corollary 3.92
- Corollary 3.93: statement only.

- Lecture 19: 02/05 (exercise Class)
  - Sheet 4, Ex 4 about the Lie algebra of the product of two Lie groups.
  - Sheet 4, Ex 2, Proving that the unitary group $\mathrm{U}(n)$ is a Lie group and calculation of its Lie algebra.
  - Lie groups have no small subgroups.

- Lecture 20: 08/05
  - Cartan's Theorem 3.94.
  - Brief discussion about idea of the proof, Lemma 3.95
  - Corollary 3.96: characterization of the Lie algebra of a closed subgroup
  - Corollary 3.97: Lie algebra of the kernel of homomorphism
  - Definition 3.98 (Ideals) and brief discussion
  - Definition 3.99: representations of Lie groups and Lie algebras
  - Remark 3.100: repr of Lie group induces repr. of Lie algebra
  - Definition of Stabilizer of vector and vector subspace
  - Proposition 3.101: Lie algebra of the stabilizer. Sketch of the proof of the first part
  - Definition of adjoint representation
  - Fundamental relation for the adjoint representation
  - adjoint representation for $\mathrm{GL}(\mathrm{n}, \mathbb{R})$.
  - Adjoint representation of a Lie algebra
  - Theorem 3.103: the induced representation of the adjoint repr. of a Lie group on the Lie algebra is the adj. repr of the Lie algebra
  - Corollary 3.104: ideals and normal subgroups. Proof of 2).
  - Theorem 3.105: center of a Lie group is the kernel of the adj. repr. No proof.

- Lecture 21: 15/05
  - Definition 4.1: solvable group
  - Definition 4.2: derived series
  - Lemma 4.3
  - Lemma 4.4: characterization of solvable groups in terms of the derived series
  - Definition 4.5: solvability length
  - Lemma 4.6
  - Proposition 4.7: refinements for (connected) topological Hausdorff groups
  - Exercise 4.8: connectedness criterion

- Lemma 4.9
- Corollary 4.10: refinement for connected Lie groups; sketch of the proof
- Example 4.11
- Statement of Theorem 4.12 (first Lie's theorem)
- Definition 4.13: weight
- Statement of Theorem 4.15: existence of weight
- Lemma 4.16: invariance of weight space

- Lecture 22: 16/05
  - Proof of Theorem 4.15
  - Proof of Theorem 4.12
  - Definition 4.17: solvable Lie algebra
  - Example 4.18 of solvable Lie algebra
  - Definition 4.19: derived series of Lie algebra
  - Definition 4.20: Characteristic ideal
  - Lemma 4.21
  - Corollary 4.22
  - Corollary 4.23, without proof
  - Lemma 4.24: characterization of solvable Lie algebras in terms of derived series
  - Definition 4.25: solvability length
  - Example 4.26 of solvable Lie algebra
  - Lemma 4.27 and Corollary 4.28, without proof
  - Theorem 4.29: characterization of connected solvable Lie groups in terms of the Lie algebra
  - Theorem 4.30: Lie group structure and Lie algebra for quotients, no proof
  - Proof of Theorem 4.29
  - Definition 4.31: solvable Lie group
  - Theorem 4.32, no proof.