Introduction to Lie groups, diary of lectures

May 30, 2024

Important note: unless otherwise stated, the proofs of Theorems, Propositions, Lemmas, and Corollaries are part of the program (if they were discussed in class).

- Lecture 1: 21/02
 - Brief historical introduction to Lie groups; Hilbert's fifth problem; Hilbert-Smith conjecture;
 - Structure of the course;
 - Definition of Topological group and elementary consequences;
 - Examples of topological groups; $GL(n, \mathbb{R})$, Iso(X), Diff(M).
 - Matrix groups: $O(n,\mathbb{R})$, A, N, O(p,q), $GL(n,\mathbb{C})$, U(n), $SL(n,\mathbb{R})$, SO(n), SO(p,q), SU(n).
- Lecture 2: 22/02
 - Compactness and local compactness;
 - Examples of compact and locally compact topological groups;
 - O(n) is compact, O(p,q) for $p,q \ge 1$ is not;
 - Statement of Proposition 2.31 about general properties of topological groups.
 - Statement and proof of Lemma 2.32 (existence of "nice" neighborhoods of the identity).
 - Proof of Prop. 2.31 (up to item 4).
- Lecture 3: 28/02
 - Proof of item 5 of Prop. 2.31;
 - The set of connected components and its group structure; examples.
 - Existence of the universal covering for locally path-connected semi locally simply connected top. spaces (statement only);
 - the fundamental group of a locally path connected and semi locally simply connected top group is abelian.
 - Local homomorphisms; definition and examples.
 - Theorem 2.42: extension of local homomorphisms from simply connected top. groups. Statement and proof up to the end of Step 1.
- Lecture 4: 29/02 (Exercise class)
 - Intuition for the compact-open topology.

- On metric spaces the compact-open topology coincides with uniform convergence on compact sets.
- An example of a topological space X, such that Homeo(X) is not a topological group. (using Cantor set)
- If X compact, then Homeo(X) is a topological group.
- Lecture 5: 06/03
 - Completion of the proof of Prop. 2.31
 - Statement of Corollary 2.38
 - Introduction to Haar measures, statement of the existence and uniqueness theorem for the Lebesgue measure
 - left group actions, continuous actions
 - endomorphism of $C_c(X)$ induced by *G*-action;
 - Statement of Riesz representation theorem.
 - Induced action on linear functionals
 - Definition of left Haar functional and measure
 - Statement of Haar's theorem (Thm. 2.42).
 - Relationship between left and right Haar measure, Lemma 2.44 and Corollary 2.45.
 - σ -finiteness of Haar measure on locally compact Hausdorff connected groups (Lemma 2.46).
- Lecture 6: 07/03
 - Statement and proof of Lemma 2.47 (general properties of Haar measures)
 - Proof of uniqueness part in Haar's theorem
 - Examples of Haar measures: Lebesgue, Haar measure on $\mathbb{R}_{>0}$, counting measure on discrete groups.
 - Mentioned existence of left Haar measures that are not right Haar measures. Left Haar=right Haar does not imply abelian.
 - The group Aut(G); left action of Aut(G) on $C_c(G)$
 - Statement and proof of Lemma 2.53; definition of mod_G .
 - Statement and proof of Lemma 2.54: $mod_G : Aut(G) \to \mathbb{R}_{>0}$ is a homomorphism.
 - Inner automorphisms, construction of modular function $\Delta_G: G \to \mathbb{R}_{>0}$.
 - Modular function measures how non-right invariant a left invariant measure is.
 - Statement of Prop. 2.55
 - Definition of left/right uniformly continuous function (Def. 2.57)
 - Statement and proof of Lemma 2.57, any $f \in C_c(G)$ is both left and right unif. continuous.
- Lecture 7: 13/03
 - Corollary 2.58, statement and proof
 - Proof of Proposition 2.55

- Definition of unimodular group and examples
- Corollary 2.61, statement and proof
- Prop. 2.62, statement and proof
- Further examples of unimodular groups (abelian and compact, Ex. 2.63)
- Introduction to homogeneous spaces.
- Topological properties, Prop. 2.64, statement and proof.
- Examples of homogeneous spaces, Ex. 2.65: S^n and GO_k .
- Lecture 8: 14/03 (Exercise class)
 - Sheet 1, Exercise 4 about $GL(n, \mathbb{R})$. One takaway is the method of showing that something is closed by viewing it as an intersection of preimages of a closed set.
 - Sheet 1, Exercise 5 about Iso(X) being a compact group, when X is a compact metric space.
 - Sheet 1, Exercise 7: topological coverings of topological groups are topological groups.
 - Fundamental group of a topological group is abelian, so $\mathbb{R}^2 \setminus \{p_1, p_2\}$ does not admit a topological group structure.
 - Examples of obtaining new groups from given ones. a) Quotienting by a normal subgroup, b) taking covers, in particular universal covers.
 - Excursion about the appearance of the universal covering of $SL(2, \mathbb{R})$ in the Thurston geometrization conjecture as one of the model geometries.
- Lecture 9: 20/03
 - Remark 2.71: the stabilizer might depend on the point; if the group acts transitively then stabilizers of different points are conjugated; the stabilizer of a point might be not normal; G/G_x might not admit any topological group structure; in general $G/G_x \times G_x$ is not homeomorphic to G.
 - Proposition 2.73: compact subgroups of $GL(n, \mathbb{R})$ are conjugated to subgroups of $O(n, \mathbb{R})$.
 - Definition 2.66 (Lattice).
 - Discussion about the definition of lattice, see Exercise 2.72.
 - Theorem 2.68, statement only.
 - Discussion on the well-posedness of Weil's formula.
 - Proposition 2.70: group with lattice must be unimodular.
- Lecture 10: 21/03
 - More details about the proof of Prop 2.70
 - Introduction to the chapter about Lie groups;
 - Definition 3.1 (Lie group)
 - Definition 3.2 (Top. manifold), Definition 3.3 (smooth structure)
 - Remark 3.4 (Top manifolds are paracompact and have countably many connected components)
 - Definition of smooth map between smooth manifolds

- Examples 3.5, 3.6. 3.7 of Lie groups
- Example 3.8 (Homeo(X) is not Lie in general) and Example 3.9 (Iso(X) might or might not be Lie)
- Definition 3.10 (Regular submanifold).
- Theorem 3.11 (statement only). Exercise 3.12 (statement).
- Theorem 3.13 (statement only)
- First part of Example 3.14: $SL(n, \mathbb{R})$ is a Lie group
- Lecture 11: 27/03
 - Second part of Example 3.14: $\mathrm{O}(n,\mathbb{R})$ is a Lie group
 - Definition 3.16 (tangent vectors)
 - Definition 3.18 (vector field and smooth vector field)
 - Discussion about the expression of vector fields in local coordinates
 - Definition 3.19 (derivation of an algebra)
 - Proposition 3.20 (Isomorphism between $Vect^{\infty}(M)$ and $Der(C^{\infty}(M))$)
 - Remark 3.21: representing germ with global smooth function and vice versa
 - Remark: composition of derivations is not a derivation in general
 - Lemma 3.22
 - Definition 3.23: bracket of vector fields
 - Definition 3.24: bracket of endomorphisms.
- Lecture 12: 28/03 (Exercise class)
 - Sheet 2, Exercise 2c) Haar measure on $GL(n, \mathbb{R})$
 - Sheet 2, Exercise 2d) Haar measure on $SL(n, \mathbb{R})$
 - Recap lattices. Fundamental domains. Examples
 - The modular group $SL(2, \mathbb{Z})$ as a lattice that is not cocompact, via its action on the hyperbolic plane $\mathbb{H} = SL(2, \mathbb{R})/SO(2)$.
- Lecture 13: 10/04
 - Properties of the bracket of endomorphisms
 - Remark 3.25: Jacobi identity is a substitute of associativity
 - Definition 3.26 (Lie algebra)
 - Example 3.27 (examples of Lie algebras)
 - Definition 3.28 (Lie algebra homomorphism)
 - Recap about definition of differential of a smooth map
 - Discussion about how to use a smooth map to "push-forward" vector fields
 - Definition 3.28 (φ -related vector fields)
 - Lemma 3.30, algebraic characterization of φ -related vector fields (no proof);
 - Proposition 3.31: brackets of φ -related vector fields are φ -related
 - Definition of push-forward of vector field with a diffeomorphism

- Corollary 3.32: diffeomorphism induces Lie algebra isomorphism between spaces of vector fields
- Recap: useful identifications for tangent spaces of vector spaces and differentials of linear maps
- Definition 3.33: smooth action, induced left translation diffeomorphisms
- Definition 3.34: G-invariant vector fields
- Definition 3.35: Lie subalgebra, left invariant vector fields are a Lie subalgebra
- Lemma 3.36
- Definition 3.37: Lie algebra of a Lie group
- Proposition 3.38, about the Lie algebra of $GL(n, \mathbb{R})$
- Lecture 14: 11/04
 - Proof of Proposition 3.38
 - Proposition 3.39: differential of Lie groups homomorphism is Lie algebra homomorphism
 - Corollary 3.40
 - Example 3.41 of Lie algebras of matrix groups
 - Example 3.43: Lie algebra of product of Lie groups
 - Broad discussion about the functor from Lie groups to Lie algebras and what comes next.
 - Examples of non-isomorphic Lie groups with isomorphic Lie algebras
 - Discussion about Cartan's theorem
 - The matrix exponential map, Proposition 3.44.
 - Sketch of proof of 1) and 2). Proof of 3).
- Lecture 15: 17/04
 - Proof of Proposition 3.44 part 4.
 - Definition 3.45, integral curve.
 - Theorem 3.46, existence and uniqueness of maximal integral curves of smooth vector fields (no proof)
 - Definition 3.47: complete vector field
 - Proposition 3.48: group law for the flow map
 - Discussion about rate of change of vector fields on smooth manifolds
 - Definition 3.49: Lie derivative
 - Theorem 3.50: Lie derivative = Lie bracket (no proof)
 - Proposition 3.51: flows commute iff vector fields commute (no proof)
 - Proposition 3.52: flows of left invariant vector fields on Lie groups
 - Definition 3.53: one parameter group
 - Corollary 3.54: one parameter groups are all obtained via Prop. 3.52
 - Definition 3.56: exponential map
 - Corollary 3.57: statement only.

- Lecture 16: 18/04 (exercise Class)
 - Sheet 3, Ex 1 about lattices.
 - Sheet 3, Ex 2, Lie subgroups that are regular submanifolds are closed.
 - Sheet 3, Ex 3 about tangent space being isomorphic to \mathbb{R}^n .
 - various notions of tangent space, including one about paths through p.
 - $-\ S^2$ is not a Lie group.
- Lecture 17: 24/04
 - Lemma 3.58: differential of multiplication.
 - Proof of Corollary 3.57: properties of the exponential.
 - Proposition 3.59: exponential and Lie group homomorphisms
 - Corollary 3.61: exponential as a preferred chart near to e.
 - Discussion about surjectivity of \exp_G : connectedness is necessary, Cartan's theorem 3.62 (statement only), example of connected Lie group with non-surjective exponential (Example 3.64).
 - Definition 3.65: abelian Lie algebra
 - Proposition 3.66: structure of abelian Lie groups
 - Introduction to the correspondence between Lie subalgebras and Lie subgroups
 - Definition 3.72: immersion, immersed vs embedded submanifolds
 - Example 3.73: immersions and immersed sumanifolds
 - Example 3.75: some Lie subgroups of T^2
 - Definition 3.76: Lie subgroup.
- Lecture 18: 25/04
 - Theorem 3.77: Lie subalgebras-Lie subgroups correspondence
 - Remark 3.78: about the uniqueness part of Thm 3.77
 - Introductory discussion about distributions and integral submanifolds
 - Definition 3.80: distribution, involutive distribution, integral submanifold, completely integrable distribution
 - Example 3.81: distributions and integral submanifolds
 - Theorem 3.82: Frobenius
 - Remark 3.83: the 1 dimensional case
 - Definition 3.84: maximal integral submanifold
 - Theorem 3.85: esistence and uniqueness of maximal integral submanifolds through each point for involutive distributions
 - Sketch of proof of Theorem 3.77
 - Theorem 3.87
 - Example 3.88: Lie algebra homomorphism not induced by Lie group homomorphism
 - Theorem 3.89: Li algebra homomorphism is induced by local Lie group homomorphism

- Lemma 3.90 (no proof)
- Proof of Theorem 3.89
- Theorem 3.91: Ado's theorem
- Corollary 3.92
- Corollary 3.93: statement only.
- Lecture 19: 02/05 (exercise Class)
 - Sheet 4, Ex 4 about the Lie algebra of the product of two Lie groups.
 - Sheet 4, Ex 2, Proving that the unitary group U(n) is a Lie group and calculation of its Lie algebra.
 - Lie groups have no small subgroups.
- Lecture 20: 08/05
 - Cartan's Theorem 3.94.
 - Brief discussion about idea of the proof, Lemma 3.95
 - Corollary 3.96: characterization of the Lie algebra of a closed subgroup
 - Corollary 3.97: Lie algebra of the kernel of homomorphism
 - Definition 3.98 (Ideals) and brief discussion
 - Definition 3.99: representations of Lie groups and Lie algebras
 - Remark 3.100: repr of Lie group induces repr. of Lie algebra
 - Definition of Stabilizer of vector and vector subspace
 - Proposition 3.101: Lie algebra of the stabilizer. Sketch of the proof of the first part
 - Definition of adjoint representation
 - Fundamental relation for the adjoint representation
 - adjoint representation for $GL(n, \mathbb{R})$.
 - Adjoint representation of a Lie algebra
 - Theorem 3.103: the induced representation of the adjoint repr. of a Lie group on the Lie algebra is the adj. repr of the Lie algebra
 - Corollary 3.104: ideals and normal subgroups. Proof of 2).
 - Theorem 3.105: center of a Lie group is the kernel of the adj. repr. No proof.
- Lecture 21: 15/05
 - Definition 4.1: solvable group
 - Definition 4.2: derived series
 - Lemma 4.3
 - Lemma 4.4: characterization of solvable groups in terms of the derived series
 - Definition 4.5: solvability length
 - -Lemma 4.6
 - Proposition 4.7: refinements for (connected) topological Hausdorff groups
 - Exercise 4.8: connectedness criterion

- -Lemma 4.9
- Corollary 4.10: refinement for connected Lie groups; sketch of the proof
- Example 4.11
- Statement of Theorem 4.12 (first Lie's theorem)
- Definition 4.13: weight
- Statement of Theorem 4.15: existence of weight
- Lemma 4.16: invariance of weight space
- Lecture 22: 16/05
 - Proof of Theorem 4.15
 - Proof of Theorem 4.12
 - Definition 4.17: solvable Lie algebra
 - Example 4.18 of solvable Lie algebra
 - Definition 4.19: derived series of Lie algebra
 - Definition 4.20: Characteristic ideal
 - Lemma 4.21
 - Corollary 4.22
 - Corollary 4.23, without proof
 - Lemma 4.24: characterization of solvable Lie algebras in terms of derived series
 - Definition 4.25: solvability length
 - Example 4.26 of solvable Lie algebra
 - Lemma 4.27 and Corollary 4.28, without proof
 - Theorem 4.29: characterization of connected solvable Lie groups in terms of the Lie algebra
 - Theorem 4.30: Lie group structure and Lie algebra for quotients, no proof
 - Proof of Theorem 4.29
 - Definition 4.31: solvable Lie group
 - Theorem 4.32, no proof.
- Lecture 23: 22/05
 - Definition 4.33: nilpotent Lie algebra
 - Definition 4.34: central series of a Lie algebra
 - Proposition 4.35: equivalent characterizations of nilpotent Lie algebras. Sketch of proof of 2) iff 3)
 - Definition 4.36: nilpotency length
 - Example 4.37
 - Theorem 4.38
 - Remark 4.39: nilpotent Lie algebras have non-trivial center
 - Lemma 4.40: statement only
 - Remark 4.41: on the assumptions in Lemma 4.40 2)

- Proof of the implication from $[\mathfrak{g},\mathfrak{g}]$ being nilpotent to \mathfrak{g} being solvable.
- Example 4.42: about the strictly upper triangular form
- Theorem 4.43: Engel's theorem, no proof
- Corollary 4.44: Lie algebra \mathfrak{g} nilpotent iff $\mathrm{ad}(\mathfrak{g})$ strictly upper triangular
- Definition 4.45 and Remark 4.46 about commutator subgroups
- Definition 4.47: nilpotent group
- Definition 4.48: descending central series
- Lemma 4.49: equivalent characterization of nilpotent groups, no proof
- Theorem 4.50: equivalent characterizations of connected nilpotent Lie groups
- Definition 4.51: Killing form
- Proposition 4.52: invariance property of the Killing form
- Exercise 4.53: further invariance properties of the Killing form.
- Lecture 24: 23/05 (Exercise class)
 - Sheet 5, Exercise 2, the classification of connected abelian Lie groups.
 - solvable groups are exactly those groups that are obtained by (repeated) extension from abelian groups
 - Recap of structure theory of nilpotent and solvable Lie groups. (Lie and Engel's theorem)
 - Sheet 5, Exercise 4, (1) and (2): Continuous group homomorphisms are automatically smooth.
 - The smooth structure of the Lie group $(\mathbb{R}^4, +)$ is never exotic.
- Lecture 25: 29/05
 - Statement of Cartan's criterion Theorem 4.54
 - -Lemma 4.55
 - Proof of the implication from \mathfrak{g} solvable to $K_{\mathfrak{g}}|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} \equiv 0$
 - Definition 4.56: (semi)simple Lie algebras and Lie groups
 - Remark 4.57: simple abstract groups vs simple Lie groups
 - Statement of Theorem 4.58
 - Lemma 4.59: orthogonal of ideal is ideal
 - Proof of the implication from \mathfrak{g} semisimple to $K_{\mathfrak{g}}$ non-degenerate
 - Statement of Theorem 4.60: criterion for semisimplicity
 - Example 4.61: semisimple Lie algebras
 - Proposition 4.62: existence and uniqueness of solvable radical
 - Definition 4.63 of solvable radical
 - -Lemma 4.64, no proof
 - Proof of Proposition 4.62
- Lecture 26: 30/05 (Exercise class)
 - Recap Adjoint representations

- Sheet 6, Exercise 2: For a connected Lie group the center is equal to the kernel of the adjoint representation.
- Example: $\operatorname{Ad}(\operatorname{SL}(2,\mathbb{R}))$.
- Repcap (semi)simple Lie algebras and groups.
- Classification of simple complex Lie algebras.