Exercise Sheet 1

1. Let A be a unital algebra and $x \in A$. Suppose there are $y, z \in A$ with $xy = zx = e$. Prove $y = z$. Moreover, prove that if there is $y' \in A$ with $xy' = e$ then $y' = y$.

2. Let X be a locally compact Hausdorff space, which is not compact. Denote by αX the one-point compactification of X. Show that the unital algebra $C_0(X)_I$ is canonically isomorphic to $C(\alpha X)$.

3. Construct a function $f \in L^1(\mathbb{R})$ such that $\|f * f^*\|_1 < \|f\|_1^2$.

   Hint: Any such function has to change the sign at some point.

4. Let A be a unital $\mathbb{C}$-algebra with a norm $\|\cdot\|$ such that:

   (a) The pair $(A, \|\cdot\|)$ is a Banach space.

   (b) The multiplication map $A \times A \to A$ is continuous in each variable.

   Show that there is an equivalent norm $\|\cdot\|_{\text{new}}$ on A such that

$$\|xy\|_{\text{new}} \leqslant \|x\|_{\text{new}} \|y\|_{\text{new}}.$$

   Hint: For each $x \in A$ define the map $R_x(z) := xz$ and put $||x||_{\text{new}} := \|R_x\|$.