Exercise Sheet 1 - Solutions

1. Let A be a unital algebra and $x \in A$. Suppose there are $y, z \in A$ with $xy = zx = e$. Prove $y = z$. Moreover, prove that if there is $y' \in A$ with $xy' = e$ then $y' = y$.

   Solution: The associativity of the multiplication in A implies
   $$y = ey = (zx)y = z(xy) = ze = z.$$
   Suppose there is another element with $xy' = y'x = e$. Then
   $$y = ye = y(xy') = (yx)y' = ey' = y'.$$

2. Let X be a locally compact Hausdorff space, which is not compact. Denote by αX the one-point compactification of X. Show that the unital algebra $C_0(X)_I$ is canonically isomorphic to $C(\alpha X)$.

   Solution: We construct an algebra morphism
   $$\Phi \colon C_0(X)_I \to C(\alpha X).$$
   with the mapping
   $$\Phi(f, \lambda)(x) := \begin{cases} f(x) + \lambda & \text{if } x \in X \\ \lambda & \text{if } x = \infty. \end{cases}$$
   Let $(f, \lambda) \in C_0(X)_I$. The function $\Phi(f, \lambda)$ is continuous on $X \subset \alpha X$ because $\Phi(f, \lambda)|_X = f$ and the subspace topology on the open subset $X \subset \alpha X$ coincides with the topology on X. So we only have to prove that $\Phi(f, \lambda)$ is continuous at ∞. Let $\epsilon > 0$ then there exists a compact $K \subset X$ such that $K \neq X$ (because X is not compact) and $|f(x)| < \epsilon$ for all $x \notin K$. This implies $|\Phi(f, \lambda)(x) - \lambda| < \epsilon$ for all $x \in \alpha X - K$. Since $\alpha X - K$ is an open non-empty neighborhood of ∞ and ϵ was arbitrary this implies that $\Phi(f, \lambda)$ is continuous. Therefore the map Φ is well-defined.

   We construct a converse
   $$\Psi \colon C(\alpha X) \to C_0(X)_I$$
   by putting
   $$\Psi(f) = (f|_X - f(\infty), f(\infty)).$$
   This map is well-defined because the subspace topology on $X \subset \alpha X$ coincides with the topology on X. We leave it to the reader to check that Φ and Ψ are unital algebra morphisms.

   Let $(f, \lambda) \in C_0(X)_I$ then
   $$\Psi(\Phi(f, \lambda))(x) = ((f + \lambda) - \lambda, \lambda) = (f, \lambda).$$
   Let $f \in C(\alpha X)$ then
   $$\Phi(\Psi(f))(x) = \Phi(f|_X - f(\infty), f(\infty))(x) = \begin{cases} f(x) & \text{if } x \in X \\ f(\infty) & \text{if } x = \infty. \end{cases}$$
   Thus Ψ and Φ are inverse to one another.

3. Construct a function $f \in L^1(\mathbb{R})$ such that $\|f * f^*\|_1 < \|f\|_1^2$.

   Hint: Any such function has to change the sign at some point.

   Solution: Consider the function $f := -1_{[-1,0]} + 1_{[0,1]}$. Then $\|f\|_1 = 2$ and

$$-(f * f^*)(x) = \begin{cases} 0 & \text{if } x \notin [-2,2] \\ x+2 & \text{if } -2 \leqslant x \leqslant -1 \\ -2x-1 & \text{if } -1 \leqslant x \leqslant 0 \\ 2x-1 & \text{if } 0 \leqslant x \leqslant 1 \\ -x+2 & \text{if } 1 \leqslant x \leqslant 2. \end{cases}$$

   To understand this example, it makes sense to compute the convolution of two indicator functions of compact, connected intervals and to draw the graph of the convolution. The above formula implies

$$\|f * f^*\|_1 = 2 < 4 = \|f\|_1^2.$$

4. Let A be a unital $\mathbb{C}$-algebra with a norm $\|\cdot\|$ such that:

   (a) The pair $(A, \|\cdot\|)$ is a Banach space.

   (b) The multiplication map $A \times A \to A$ is continuous in each variable.

   Show that there is an equivalent norm $\|\cdot\|_{\text{new}}$ on A such that

$$\|xy\|_{\text{new}} \leqslant \|x\|_{\text{new}}\|y\|_{\text{new}}.$$

   Hint: For each $x \in A$ define the map $R_x(z) := xz$ and put $||x||_{\text{new}} := \|R_x\|$.

   Solution: The definition of $\|\cdot\|_{\text{new}}$ is well-defined because the multiplication map is continuous in the second variable. For all $x, y \in A$ we get $R_{xy} = R_x R_y$ because associativity implies

$$R_{xy}(z) = (xy)z = x(yz) = R_x(yz) = R_x(R_y(z)) = (R_x R_y)(z)$$

   for all $z \in A$. This implies

$$\|xy\|_{\text{new}} = \|R_{xy}\| = \|R_x R_y\| \leqslant \|R_x\|\|R_y\| = \|x\|_{\text{new}}\|y\|_{\text{new}}.$$

   Suppose $\|x\|_{\text{new}} = 0$ then we get $0 = R_x(1) = x$ and hence $x = 0$. The distributive law implies

$$R_{x+y}(z) = (x+y)z$$

   for all $z \in A$ so we get

$$\|x+y\|_{\text{new}} = \|R_{x+y}\| = \|R_x + R_y\| \leqslant \|R_x\| + \|R_y\| = \|x\|_{\text{new}} + \|y\|_{\text{new}}.$$

   Note that for each $\lambda \in \mathbb{C}$ and $x \in X$ we have

$$R_{\lambda x}(z) = (\lambda x)z = \lambda(xz) = \lambda R_x(z).$$

We get

$$\|\lambda x\|_{\text{new}} = \|R_{\lambda x}\| = \|\lambda R_x\| = |\lambda|\|R_x\|.$$

Thus $\|\cdot\|_{\text{new}}$ is a submultiplicative norm on A.

Each $x \in A$ satisfies

$$\|x\|_{\text{new}} = \sup_{\|y\|\leqslant 1} \|xy\| \geqslant \|x\|$$

So the open mapping theorem implies that the norms $\|\cdot\|$ and $\|\cdot\|_{\text{new}}$ are equivalent.