Exercise Sheet 2 - Solutions

1. Let X be a locally compact Hausdorff space. Compute the spectrum $\mathrm{Sp}_{C_0(X)}(f)$ for each $f \in C_0(X)$.

   Solution: Suppose X is compact and let $f \in C(X)(= C_0(X))$. We claim that f is invertible if and only if $f(x) \neq 0$ for all $x \in X$. Suppose f is invertible. Then there exists $g \in C(X)$ with $fg = e$. The unit in $C(X)$ is the constant function with value 1, so this implies $f(x)g(x) = 1$ for all $x \in X$. This equality implies $f(x) \neq 0$ for all $x \in X$.

   Suppose $f(x) \neq 0$ for all $x \in X$. Then f defines a continuous map
$$f\colon X \to \mathbb{C}^*$$
   where we consider $\mathbb{C}^* \subset \mathbb{C}$ with the subspace topology. The inversion map
$$\mathrm{inv}\colon \mathbb{C}^* \to \mathbb{C}^*,\ z \mapsto z^{-1}$$
   is holomorphic, so it is continuous. Define the continuous function $g := \mathrm{inv} \circ f$. Then
$$f(x)g(x) = f(x)\mathrm{inv}(f(x)) = 1.$$
   Thus $fg = e$, so f is invertible.

   We can apply the claim to obtain
$$\mathrm{Sp}_{C(X)}(f) = \{\lambda \in \mathbb{C} \mid f - \lambda e \text{ not invertible}\} = \{\lambda \in \mathbb{C} \mid \exists x \in X : f(x) - \lambda = 0\} = \mathrm{im}(f).$$
   Suppose X is not compact. The algebra $C_0(X)$ is not unital, so for each $f \in C_0(X)$ we have defined
$$\mathrm{Sp}_{C_0(X)}(f) := \mathrm{Sp}_{(C_0(X))_I}(f).$$
   We use the isomorphism $\phi : (C_0(X))_I \cong C(\alpha X)$ where αX denotes the one-point compactification of X to write
$$\mathrm{Sp}_{(C_0(X))_I}(f) = \mathrm{Sp}_{C(\alpha X)}(\phi(f)) = \mathrm{im}(\phi(f)).$$
   I claim
$$\mathrm{im}(\phi(f)) = \overline{\mathrm{im}(f)}$$
   for all $f \in C_0(X)$. Indeed, note that $\mathrm{im}(\phi(f))$ is compact because αX is compact. This implies that $\mathrm{im}(\phi(f))$ is closed in $\mathbb{R}$ and hence $\overline{\mathrm{im}(f)} \subset \mathrm{im}(\phi(f))$. For the other direction, let $(x_n)_n \in X$ be a sequence converging to ∞ in αX. Then
$$\phi(f)(\infty) = \lim_{n\to\infty} \phi(f)(x_n) = \lim_{n\to\infty} f(x_n).$$
   This gives $\phi(f)(\infty) \in \overline{\mathrm{im}(f)}$. Thus we get
$$\mathrm{im}(\phi(f)) = \mathrm{im}(f) \cup \{\phi(f)(\infty)\} \subset \overline{\mathrm{im}(f)}.$$
   We arrive at
$$\mathrm{Sp}_{C_0(X)}(f) = \overline{\mathrm{im}(f)}$$
   for all $f \in C_0(X)$.

2. Let Γ be an abelian group and $A = \ell^1(\Gamma)$ with convolution product. Compute the Guelfand spectrum $\widehat{A}$ of A as a set.

Solution: Let $\Gamma^\wedge$ be the set of group morphism $\Gamma \to \mathbb{T}$, where $\mathbb{T} := \{z \in \mathbb{C}^* : |z| = 1\}$. We construct a bijection

$$\Phi : \widehat{A} \to \Gamma^\wedge,\ \varphi \mapsto (\gamma \mapsto \varphi(\delta_\gamma))$$
$$\Psi : \Gamma^\wedge \to \widehat{A},\ \chi \mapsto \Big(f \mapsto \sum_{\gamma \in \Gamma} f(\gamma)\chi(\gamma)\Big).$$

Let $\varphi \in \widehat{A}$. For all $\gamma, \mu \in \Gamma$, we have

$$\Phi(\varphi)(\gamma\mu) = \varphi(\delta_{\gamma\mu}) = \varphi(\delta_\gamma)\varphi(\delta_\mu) = \Phi(\varphi)(\gamma)\Phi(\varphi)(\mu).$$

So the function $\Phi(\varphi)$ is multiplicative.

Suppose $\varphi(\delta_1) = 0$. Then $\varphi(\delta_a) = \varphi(\delta_a * \delta_1) = \varphi(\delta_a)\varphi(\delta_1) = 0$. This implies that φ vanishes at all functions with finite support. These are dense in A, so this implies $\varphi = 0$ by continuity of φ. This is a contradiction to φ lying in the Guelfand spectrum, so we must have $\varphi(\delta_1) \neq 0$. The multiplicativity property implies

$$\Phi(\varphi)(1) = \Phi(\varphi)(1 \cdot 1) = \Phi(\varphi)(1)^2.$$

Thus $\Phi(\varphi)(1) = 1$. In particular,

$$1 = \Phi(\varphi)(1) = \Phi(\varphi)(\gamma\gamma^{-1}) = \Phi(\varphi)(\gamma)\Phi(\varphi)(\gamma^{-1})$$

for each $\gamma \in \Gamma$. Thus we get $\Phi(\varphi)(\gamma) \neq 0$ and $\Phi(\varphi)(\gamma^{-1}) = \Phi(\varphi)(\gamma)^{-1}$ for each $\gamma \in \Gamma$.

Suppose there is $\gamma \in \Gamma$ with $|\Phi(\varphi)(\gamma)| \neq 1$. We either have $|\Phi(\varphi)(\gamma)| > 1$ or $|\Phi(\varphi)(\gamma^{-1})| > 1$, so we can assume $|\Phi(\varphi)(\gamma)| > 1$ without loss of generality. Pick a real number $c \in \mathbb{R}$ with

$$|\Phi(\varphi)(\gamma)| > c > 1.$$

The functions

$$f_n := c^{-n}\delta_n$$

converge to 0 in $\ell^1(\Gamma)$ as $n \to \infty$ because $0 < c^{-1} < 1$. Thus

$$0 = \lim_{n\to\infty} \varphi(f_n) = \lim_{n\to\infty} c^{-n}\Phi(\varphi)(\gamma)^n.$$

This is a contradiction because $|c^{-n}\Phi(\varphi)(\gamma)^n| > 1$. Thus we get $|\Phi(\varphi)(\gamma)| = 1$ for all $\gamma \in \Gamma$. This implies that Φ is well-defined.

Let $\chi \in \Gamma^\wedge$. Note that $\Psi(\chi)$ is a continuous functional (because $|\Psi(\chi)(f)| \leqslant \|f\|$). Consider $\gamma, \mu \in \Gamma$ then we have

$$\Psi(\chi)(\delta_\gamma * \delta_\mu) = \Psi(\chi)(\delta_{\gamma\mu}) = \chi(\gamma\mu) = \chi(\gamma)\chi(\mu) = \Phi(\chi)(\gamma)\Phi(\chi)(\mu).$$

This implies
$$\Psi(\chi)(f * g) = \Psi(\chi)(f)\Psi(\chi)(g)$$
for all $f, g \in \ell^1(\Gamma)$ with finite support by linearity of Ψ. The functions with finite support are dense in $\ell^1(\Gamma)$, so the continuity of Φ and the continuity of convolution imply that $\Psi(\chi)$ is an algebra morphism. Therefore the map Ψ is well-defined.

We have
$$\Phi(\Psi(\chi))(\gamma) = \Psi(\chi)(\delta_\gamma) = \chi(\gamma)$$
for each $\chi \in \Gamma^\wedge$ and $\gamma \in \Gamma$. For each $\varphi \in \ell^1(\Gamma)$, we have
$$\Psi(\Phi(\varphi))(\delta_\gamma) = \Phi(\varphi)(\gamma) = \varphi(\delta_\gamma).$$
By linearity and continuity, this implies $\Psi(\Phi(\varphi)) = \varphi$. Thus Φ and Ψ are inverse to one another.

3. Let $A = L^1([0, 1])$ be the Volterra algebra (see Example 1.6 (iv) in the notes) and $f_0 \equiv 1$. Identify the subalgebra of A generated by f_0.

   Solution: Let $g \in L^1([0, 1])$ be a function. Put
$$F(x) := f_0 * g(x) = \int_0^x g(t)dt.$$
   The Lebesgue differentiation theorem implies $F'(x) = f(x)$ for almost all $x \in [0, 1]$. Moreover, we have $F(0) = 0$. This determines the function f uniquely.

   We define recursively
$$f_{n+1} := f_0 * f_n$$
   for each $n \geqslant 0$. We claim $f_n(x) = x^n/n!$ for each $n \geqslant 0$. We prove it by induction on n. The case $n = 0$ follows from the definition of f_0. If the induction hypothesis is satisfied for some $n \in \mathbb{N}$, note that
$$\frac{d}{dx}(x^{n+1}/(n+1)!) = x^n/n!.$$
   So the uniqueness of the antiderivative implies the claimed equality.

   Because A is commutative the subalgebra generated by f_0 must be the algebra of polynomial functions $\mathbb{C}[X]$.