Exercise Sheet 3 - Solutions

1. Let X be a locally compact Hausdorff space. Show that the canonical map

$$\Phi\colon X \to \widehat{C_0(X)}$$

is a homeomorphism where $\widehat{C_0(X)}$ is endowed with the Guelfand topology.

Solution: Consider distinct $x, y \in X$. By Urysohn's Lemma, there exists $f \in C_0(X)$ with $f(x) \neq f(y)$. This implies $\Phi(x)(f) \neq \Phi(y)(f)(= f(y))$.

Consider $\varphi \in \widehat{C_0(X)}$ and let $I := \ker(\varphi)$. This ideal is maximal by Theorem 3.15, so Proposition 3.10 shows that there is a point $x \in X$ with $I = \{f \in C_0(X) : f(x) = 0\}$. By Urysohn's Lemma, there exists $\phi \in C_c(X)$ such that $\phi(x) = 1$. For each $f \in C_0(X)$, we get

$$\varphi(f) = \varphi(f(x)\phi - (f(x)\phi - f)) = \varphi(f(x)\phi) - \varphi(f(x)\phi - f) = \varphi(f(x)\phi) = f(x)\varphi(\phi).$$

Note that we have

$$\varphi(\phi)^2 = \varphi(\phi^2) = \phi^2(x)\varphi(\phi) = \varphi(\phi).$$

We have $\varphi(\phi) \neq 0$ because $\varphi \neq 0$ thus this equation implies $\varphi(\phi) = 1$. We arrive at $\varphi = \Phi(x)$. Thus Φ is surjective.

We now turn to proving that the map is a homomorphism. Consider $\varphi_0 \in \widehat{C_0(X)}, f_1, \ldots, f_n \in C_0(X)$ and $\epsilon > 0$. We have

$$\Phi^{-1}(U(\varphi_0 : a_1, \ldots, a_n; \epsilon)) = \{x \in X : |\varphi_0(f_i) - f_i(x)| < \epsilon \; 1 \leqslant i \leqslant n\}.$$

The set on the right is open because the functions f_i are continuous.

Suppose $U \subset X$ is an open. It follows from the correspondence between ideals and closed subsets (see Proposition 3.10) that there are $f_i \in C_0(X)$ such that

$$U = \bigcup_{i \in I} \{x \in X : f_i(x) \neq 0\}.$$

The map Φ is surjective, so we can write

$$\Phi(\{x \in X : f_i(x) \neq 0\}) = \{x \in \widehat{C_0(X)} : f_i(x) \neq 0\}.$$

The set on the right is open, implying that Φ is an open mapping. This implies the continuity of Φ.

2. Find an example of a commutative Banach algebra A for which the Guelfand transform $A \to C_0(\widehat{A})$ is not surjective.

Solution: Let $I \subset \mathbb{R}$ be a compact interval and consider the involutive Banach algebra

$$A := C^1([0, 1])$$

with the norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$ and the involution $f^*(t) := \overline{f(t)}$. For each $t \in I$, we have the ideal

$$J_t := \{f \in A : f(t) = 0\}.$$

This ideal is maximal because it is the kernel of the character $f \mapsto f(t)$. Suppose $J \neq J_t$ for all $t \in I$. This means there are $f_t \in J$ such that $f_t(t) \neq 0$ for each $t \in I$. Remark $ff^*(t) \geqslant 0$ for all $t \in \mathbb{R}$. The compactness of I implies that there are $t_1, \ldots, t_n \in I$ such that

$$f(x) := \sum_{i=1}^{n} |f_t(x)|^2 > 0$$

for all $x \in I$. We have $f_t f_t^* \in J$ for all $t \in I$, so $f \in J$. The function f does not vanish, so it is a unit by the quotient rule. Thus

$$1 = ff^{-1} \in J.$$

This implies $J = A$, which contradicts the maximality of J. Hence every maximal ideal is I_t for some $t \in I$.

By a similar argument as in exercise 1, where one takes a constant function for ϕ, the Gualfand dual is identified with $\widehat{A} = I$ and the Guelfand transform is

$$C^1(I) \to C(I),\ f \mapsto f.$$

If the interior of I is non-empty then there exists a function that is continuous but not differentiable. Thus the Guelfand transform is not surjective.

3. Consider the Banach algebra $A := \ell^1(\mathbb{Z})$ with convolution product and

$$B := \{f \in A : f(n) = 0\ \forall n < 0\}.$$

Show that B is a unital subalgebra of A. Moreover, prove $\mathrm{Sp}_A(\delta_1) \subsetneq \mathrm{Sp}_B(\delta_1)$.

Solution: Let $f, g \in B$. For each $n \in \mathbb{Z}$ we have

$$(f * g)(n) = \sum_{a+b=n} f(a)g(b).$$

When $n < 0$ each term in the sum on the right has either $a < 0$ or $b < 0$, so $f * g \in B$.

The evaluation functional

$$\ell^1(\mathbb{Z}) \to \mathbb{C},\ f \mapsto f(n)$$

is continuous for each $n \in \mathbb{Z}$. Thus $B \subset A$ is closed and hence complete.

Note that

$$\delta_1^{-1} * \delta_1 = \delta_0 = 1,$$

so $\delta_1^{-1} = \delta_{-1}$. Thus $0 \notin \mathrm{Sp}_A(\delta_1)$. Suppose there is $f \in B$ with $\delta_1 * f = \delta_0$. The uniqueness of the inverse implies $f = \delta_{-1}$ and hence $f \notin B$. This is a contradiction and it implies $0 \in \mathrm{Sp}_B(\delta_1)$.