Exercise Sheet 4 - Solutions

1. Let X be a non-compact LCH space and αX the one-point compactification of X. Show that the natural isomorphism from Sheet 1
$$C_0(X)_I \to C(\alpha X)$$
is norm preserving.

2. Let $\mathscr{H}$ be a complex Hilbert space and $E \subset \mathscr{H}$ a vector subspace. Show
$$(E^\perp)^\perp = \overline{E}.$$

3. Let $\mathscr{H}$ be a complex Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Prove
$$\ker(T^*) = (\mathrm{im}(T))^\perp$$
and
$$\ker(T) = (\mathrm{im}(T^*))^\perp.$$

4. Let $\mathscr{H}$ be a finite-dimensional complex Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Prove that T is normal if and only if $\mathscr{H}$ admits an orthonormal basis of eigenvectors.