Exercise Sheet 4 - Solutions

1. Let X be a non-compact LCH space and αX the one-point compactification of X. Show that the natural isomorphism from Sheet 1

$$C_0(X)_I \to C(\alpha X)$$

is norm preserving, where we equip $C_0(X)_I$ with the norm from Proposition 4.4.

Solution: We begin by proving the following claim: Let A be an involutive Banach algebra. Suppose there are continuous norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on A (a continuous norm is defined to be a norm which recovers the topology on A) which satisfy the C*-condition

$$\|xx^*\|_i = \|x\|_i^2$$

for all $x \in A$. Then $\|\cdot\|_1 = \|\cdot\|_2$.

Let $x \in A$. If x is self-adjoint, then we have

$$\|x\|_1 = \|x\|_{\mathrm{Sp}} = \|x\|_2.$$

Let $x \in A$ be arbitrary. There exists self-adjoint $x_1, x_2 \in A$ such that $x = x_1 + ix_2$. The above equation implies

$$\|x\|_1 = \|(x_1 + ix_2)(x_1 + ix_2)^*\|_1^{1/2} = \|x_1^2 + x_2^2\|_1^{1/2} = \|x_1^2 + x_2^2\|_2^{1/2} = \|x\|_2.$$

Let A, B be C*-algebras and $\Phi : A \to B$ an injective morphism of involutive Banach algebras, i.e. an injective homomorphism with $\Phi(x^*) = \Phi(x)^*$ for all $x \in A$. Then

$$\|x\|_{\text{new}} := \|\Phi(x)\|$$

is a continuous norm on A (by the open mapping theorem), which satisfies the C*-condition. Therefore, the claim implies

$$\|x\| = \|x\|_{\text{new}} = \|\Phi(x)\|$$

for all $x \in A$. Thus Φ is norm-preserving. In particular, the natural isomorphism from Sheet 1 is norm-preserving (because it intwertwines the involutions).

2. Let $\mathscr{H}$ be a complex Hilbert space and $E \subset \mathscr{H}$ a vector subspace. Show

$$(E^\perp)^\perp = \overline{E}.$$

Solution: Note that any continuous functional $\mathscr{H} \to \mathbb{C}$, which vanishes on E, automatically vanishes on $\overline{E}$ by continuity. Thus we have

$$\overline{E}^\perp = E^\perp.$$

This means it suffices to prove the claim for closed subspaces E because the claim for $\overline{E}$ implies the theorem for E.

Assume E is a closed subspace. The inclusion

$$E \subset (E^{\perp})^{\perp}$$

can be proven by unwrapping the definitions. Let $v \in (E^{\perp})^{\perp}$. Because E is a closed subspace of a Hilbert space, there are unique vectors $e_0 \in E$ and $e_1 \in E^{\perp}$ such that $v = e_0 + e_1$. We have

$$0 = \langle v, e_1 \rangle = \|e_1\|^2.$$

Hence $v \in E$.

3. Let $\mathscr{H}$ be a complex Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Prove

$$\ker(T^*) = (\operatorname{im}(T))^{\perp}$$

and

$$\ker(T) = (\operatorname{im}(T^*))^{\perp}.$$

Solution: We only give the proof of the first equality because the second equality follows from the first equality and $T^{**} = T$. Let $v \in \mathscr{H}$. We have a chain of equivalences

$$\begin{aligned} v \in \ker(T^*) &\Leftrightarrow T^* v = 0 \\ &\Leftrightarrow \forall w \in \mathscr{H} : \langle T^* v, w \rangle = 0 \\ &\Leftrightarrow \forall w \in \mathscr{H} : \langle v, Tw \rangle = 0 \\ &\Leftrightarrow v \in \operatorname{im}(T). \end{aligned}$$

4. Let $\mathscr{H}$ be a finite-dimensional complex Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Prove that T is normal if and only if $\mathscr{H}$ admits an orthonormal basis of eigenvectors.

Solution: Suppose T is normal. The spectral theorem states that $\mathscr{H}$ admits an orthonormal basis of eigenvectors for T.

Suppose there is an ONB $\{e_1, \ldots, e_n\}$ of $\mathscr{H}$ such that $Te_i = \lambda_i e_i$ for $\lambda_i \in \mathbb{C}$. We have

$$\langle T^* e_i, e_j \rangle = \langle e_i, Te_j \rangle = \overline{\lambda_j} \delta_{ij}.$$

Thus $T^* e_i = \overline{\lambda_i} e_i$, so

$$T(T^* e_i) = |\lambda_i|^2 e_i$$

and

$$T^*(Te_i) = |\lambda_i|^2 e_i.$$

The vectors e_i generate $\mathscr{H}$, so

$$TT^* = T^* T.$$