Exercise Sheet 5

Let X be a compact Hausdorff space, $\mathcal{B}$ the set of Borel sets in X, $\mathscr{H}$ a Hilbert space, and

$$E\colon \mathcal{B} \to \mathcal{L}(\mathscr{H})$$

a resolution of the identity. Define

$$N := \{f \in \mathcal{B}^\infty(X) : \|f\|_\infty = 0\}.$$

1. Show that the space $\mathcal{B}^\infty(X)$ equipped with the supremum norm $\|\cdot\|$ is a C*-algebra.
2. Prove for all $f, g \in \mathcal{B}^\infty(X)$:
   (a) $\|f\|_\infty \leqslant \|f\|$.
   (b) $\|f+g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty$.
   (c) $\|fg\|_\infty \leqslant \|f\|_\infty \|g\|_\infty$.
3. Prove the equality $\|f+g\|_\infty = \|f\|_\infty$ for all $f \in \mathcal{B}^\infty(X)$ and $g \in N$. Deduce that the quotient norm on $\mathcal{B}^\infty(X)/N$ is given by $\|f+N\| = \|f\|_\infty$.
4. Show that $L^\infty(X) := \mathcal{B}^\infty(X)/N$ equipped with the quotient norm is a C*-algebra. Furthermore, show that
   $$\mathrm{Sp}_{L^\infty(X)}(f+N) = \mathrm{essim}(f)$$
   the spectrum equals the essential image for all $f \in \mathcal{B}^\infty(X)$.