Exercise Sheet 5 - Solutions

Let X be a compact Hausdorff space, $\mathcal{B}$ the set of Borel sets in X, $\mathscr{H}$ a Hilbert space, and

$$E\colon \mathcal{B} \to \mathcal{L}(\mathscr{H})$$

a resolution of the identity. Define

$$N := \{f \in \mathcal{B}^\infty(X) : \|f\|_\infty = 0\}.$$

1. Show that the space $\mathcal{B}^\infty(X)$ equipped with the supremum norm $\|\cdot\|$ is a C*-algebra.

   Solution: We only cover parts of the proof. Let $(f_n) \in \mathcal{B}^\infty(X)$ be a Cauchy sequence. By definition of the norm, the pointwise limit

   $$f(x) := \lim_{n\to\infty} f_n(x)$$

   exists. A result in measure theory shows that f is measurable. Let $\epsilon > 0$ and pick $N \geqslant 0$ such that $\|f_n - f_m\| < \epsilon$ for all $n, m \geqslant N$. This implies

   $$|f_n(x) - f(x)| = \lim_{n\to\infty} |f_n(x) - f_m(x)| \leqslant \epsilon$$

   for all $x \in X$. Hence $f_n \to f$ as $n \to \infty$.

   Let $f \in \mathcal{B}^\infty(X)$ then we have

   $$\|ff^*\| = \sup_{x\in X} |f(x)f^*(x)| = \sup_{x\in X} |f(x)|^2 = \big(\sup_{x\in X} |f(x)|\big)^2 = \|f\|^2.$$

2. Prove for all $f, g \in \mathcal{B}^\infty(X)$:

   (a) $\|f\|_\infty \leqslant \|f\|$.

   (b) $\|f + g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty$.

   (c) $\|fg\|_\infty \leqslant \|f\|_\infty \|g\|_\infty$.

   Solution: Let $f \in \mathcal{B}^\infty(X)$. I claim

   $$\|f\|_\infty = \inf\{t \geqslant 0 \mid E(f^{-1}(\{|z| > t\})) = 0\}.$$

   The inequality

   $$\|f\|_\infty \leqslant \inf\{t \geqslant 0 \mid E(f^{-1}(\{|z| > t\})) = 0\}$$

   follows from the definition of $\|f\|_\infty$. Consider $t > \|f\|_\infty$. Then $f^{-1}(\{|z| > t\}) \subset \mathrm{essIm}(f)^c$. Thus there exists a countable open cover $f^{-1}(\{|z| > t\}) = \cup_{n\in\mathbb{N}} D_n$ such that $E(D_n) = 0$ for all $n \in \mathbb{N}$. This implies $E(f^{-1}(\{|z| > t\})) = 0$ from which we get the opposite inequality.

   Let $f, g \in \mathcal{B}^\infty(X)$. Then

   $$\|f\|_\infty = \inf\{t \geqslant 0 \mid E(f^{-1}(\{|z| > t\})) = 0\} \leqslant \inf\{t \geqslant 0 \mid f^{-1}(\{|z| > t\}) = \emptyset\} = \|f\|.$$

Consider $\lambda_i > \|f_i\|_\infty$. Consider $x \in X$ such that $|f(x) + g(x)| > \lambda_1 + \lambda_2$. We either have $|f(x)| > \lambda_1$ or we have $|g(x)| > \lambda_2$. This gives

$$(f_1 + f_2)^{-1}(\{|z| > \lambda_1 + \lambda_2\}) \subset f_1^{-1}(\{|z| > \lambda_1\} \cup f_2^{-1}(\{|z| > \lambda_2\}$$

and thus

$$E((f_1 + f_2)^{-1}(\{|z| > \lambda_1 + \lambda_2\})) = 0$$

by the third property of a resolution of the identity. A similar argument proves the inequality (c).

3. Prove the equality $\|f+g\|_\infty = \|f\|_\infty$ for all $f \in \mathcal{B}^\infty(X)$ and $g \in N$. Deduce that the quotient norm on $\mathcal{B}^\infty(X)/N$ is given by $\|f + N\| = \|f\|_\infty$.

   Solution: By Exercise 2, (ii) we have

$$\|f + g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty = \|f\|_\infty.$$

   We have $-g \in N$ (by homogeneity of $\|\cdot\|_\infty$), so

$$\|f\|_\infty = \|(f + g) - g\|_\infty \leqslant \|f + g\|_\infty.$$

   The quotient norm is by definition

$$\|f + N\| = \inf_{e\in N} \|f + e\| \geqslant \inf_{e\in N} \|f + e\|_\infty = \|f\|_\infty.$$

   Define

$$e(x) := \begin{cases} -f(x) & \text{if } f(x) \notin \operatorname{essim}(f) \\ 0 & \text{else.} \end{cases}$$

   Then $e \in N$ because $E(f^{-1}(\operatorname{essim}(f))^c) = 0$. We get

$$\|f + N\| \leqslant \|f + e\| = \|f\|_\infty.$$

4. Show that $L^\infty(X) := \mathcal{B}^\infty(X)/N$ equipped with the quotient norm is a C*-algebra. Furthermore, show that

$$\operatorname{Sp}_{L^\infty(X)}(f + N) = \operatorname{essim}(f)$$

   the spectrum equals the essential image for all $f \in \mathcal{B}^\infty(X)$.

   Solution: Let $f \in \mathcal{B}^\infty(X)$ and define $a(z) := |z|^2$ for all $z \in \mathbb{C}$. Then $ff^* = a \circ f$. Each open subset $U \subset \mathbb{C}$ satisfies

$$U \subset (\operatorname{essim}(ff^*))^c \Leftrightarrow E((ff^*)^{-1}(U)) = 0 \Leftrightarrow E(f^{-1}(a^{-1}(U))) = 0 \Leftrightarrow a^{-1}(U) \subset (\operatorname{essim}(f))^c.$$

   This implies (after a small argument)

$$\operatorname{essim}(ff^*) = a(\operatorname{essim}(f))$$

and hence

$$\|ff^*\|_\infty = \sup\{|\lambda| : \lambda \in \mathrm{essim}(ff^*)\} = \sup\{|a(\lambda)| : \lambda \in \mathrm{essim}(f)\} = \sup\{|\lambda|^2 : \lambda \in \mathrm{essim}(f)\} = \|f\|_\infty^2.$$

Let $f \in \mathcal{B}^\infty(X)$ and suppose there is $g \in \mathcal{B}^\infty(X)$ such that $fg + N = 1 + N$. There is a Borel subset $C_1 \subset X$ such that $E(C_1) = 0$ and

$$f(x)g(x) = 1$$

for all $x \notin C_1$. There is a Borel subset $C_2 \subset X$ such that $|g(x)| \leqslant \|g\|_\infty$ for all $x \notin C_2$ and $E(C_2) = 0$. For all $x \notin C_1 \cup C_2$ we have

$$|f(x)| = 1/|g(x)| \geqslant 1/\|g\|_\infty.$$

Put $r := 1/(2\|g\|_\infty)$ then

$$f^{-1}(B_r(0)) \subset C_1 \cup C_2.$$

This implies $0 \notin \mathrm{essim}(f)$ because

$$E(f^{-1}(B_r(0))) = E\big(f^{-1}(B_r(0)) \cap (C_1 \cup C_2)\big) = E(f^{-1}(B_r(0)))E(C_1 \cup C_2) = 0$$

Let $f \in \mathcal{B}^\infty(X)$ and suppose $0 \notin \mathrm{essim}(f)$. The essential image is closed, so there exists $r > 0$ such that $B_r(0) \subset \mathrm{essim}(f)^c$. Let $S := \{x \in X \mid f(x) \in \mathrm{essim}(f)\}$. The set S is a Borel subset because $\mathrm{essim}(f)$ is closed and f is measurable. The definition of the essential image implies

$$E(S) = 0.$$

Define

$$g(x) := \begin{cases} 1/g(x) & \text{if } x \in S \\ 0 & \text{else.} \end{cases}$$

We have $f(x)g(x) = 1$ for all $x \in X'$. This says the function $fg - 1$ is supported on S^c. Hence $fg - 1 \in N$ and thus

$$fg + N = 1 + N.$$

We arrive at

$$0 \in \mathrm{Sp}_{L^\infty(X)}(f + N) \Leftrightarrow 0 \notin \mathrm{essim}(f)$$

for each $f \in \mathcal{B}^\infty(X)$. Note that $\lambda \in \mathrm{Sp}_{L^\infty(X)}(f + N)$ if and only if $0 \in \mathrm{Sp}_{L^\infty(X)}(f - \lambda + N)$ for all $f \in \mathcal{B}^\infty(X)$ and $\lambda \in \mathbb{C}$.