# Exercise Sheet 6

1. Let $\mathscr{H}$ be a finite-dimensional Hilbert space and let $T\colon \mathscr{H} \to \mathscr{H}$ be a self-adjoint endomorphism. Define
$$A := \{p(T) : p \in \mathbb{C}[X]\}$$
to be the (unital) sub-C*-algebra generated by $T$.
   (a) Show that there is $v \in \mathscr{H}$ with $Av = \mathscr{H}$ if and only if each eigenvalue of $T$ has multiplicity one.
   (b) Identify the Guelfand spectrum $\widehat{A}$ with the usual spectrum
$$\mathrm{Sp}(T) := \{\lambda \in \mathbb{C} \mid \exists v \in \mathscr{H} - \{0\} : Tv = \lambda v\}.$$
   (c) Denote by $\mathcal{B}$ be the Borel subsets of $\widehat{A}$. The spectral theorem shows that there is a resolution of the identity $E\colon \mathcal{B} \to \mathscr{L}(\mathscr{H})$ such that
$$T = \int_{\widehat{A}} T dE.$$
   Determine the map $E$.

2. Let $\mathscr{H}$ be a Hilbert space and $U \in \mathscr{L}(\mathscr{H})$ a unitary operator. Define the C*-algebra
$$A := \overline{\{p(U, U^*) : p \in \mathbb{C}[X, Y]\}}.$$
This algebra is commutative, so the spectral theorem implies that there is a resolution of the identity $E$ with
$$U = \int_{\widehat{A}} U dE.$$
Define the set $X := \{\chi \in \widehat{A} : \chi(U) = 1\}$.
   (a) Let $\chi \in \widehat{A}$. Prove $|\chi(U)| = 1$. Apply the formula for the geometric series to obtain
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi(U^n) = 1_X(\chi)$$
   for each $\chi \in \widehat{A}$ where $1_X$ denotes the indicator function of $X$.
   (b) Let $W := \{v \in \mathscr{H} : Uv = v\}$ and denote by $P\colon \mathscr{H} \to \mathscr{H}$ the orthogonal projection onto $W$. Prove
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i v = Pv$$
   for each $v \in \mathscr{H}$.

*Hint:* Use Exercise (a) and dominated convergence to prove

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i = \int_{\widehat{A}} 1_X \, dE = E(X).$$

This result is known as von Neumann's ergodic theorem. Applications of this theorem can be found in [Zi, Ch. 4.4].