Exercise Sheet 6 - Solutions

1. Let $\mathscr{H}$ be a finite-dimensional Hilbert space and let $T\colon \mathscr{H} \to \mathscr{H}$ be a self-adjoint endomorphism. Define
$$A := \{p(T) : p \in \mathbb{C}[X]\}$$
to be the (unital) sub-C*-algebra generated by T.

   (a) Show that there is $v \in \mathscr{H}$ with $Av = \mathscr{H}$ if and only if each eigenvalue of T has multiplicity one.

   (b) Identify the Guelfand spectrum $\widehat{A}$ with the usual spectrum
$$\mathrm{Sp}(T) := \{\lambda \in \mathbb{C} \mid \exists v \in \mathscr{H} - \{0\} : Tv = \lambda v\}.$$

   (c) Denote by $\mathcal{B}$ be the Borel subsets of $\widehat{A}$. The spectral theorem shows that there is a resolution of the identity $E\colon \mathcal{B} \to \mathscr{L}(\mathscr{H})$ such that
$$T = \int_{\widehat{A}} T dE.$$
   Determine the map E.

Solution:

(a) Suppose there is $v \in V$ such that $Av = \mathscr{H}$. By the spectral theorem, there is unique decomposition $v = \sum_{\lambda \in \mathrm{Sp}(T)} v_\lambda$ such that $Tv_\lambda = \lambda v_\lambda$. This implies that $\mathscr{H}$ is spanned by the v_λ because we have
$$P(T)v = \sum_{\lambda \in \mathrm{Sp}(T)} P(\lambda) v_\lambda$$
for each $P(X) \in \mathbb{C}[X]$. Thus there are at least $\dim(\mathscr{H})$ vectors in $\{v_\lambda : \lambda \in \mathrm{Sp}(T)\}$. This implies that there are at least $\dim(\mathscr{H})$ distinct eigenvectors hence that all eigenvalues occur with multiplicity 1.

Suppose each eigenvalue has multiplicity one. We have $1 \in A$, so by replacing T with $T + \lambda$ we can assume that all eigenvalues are non-zero. Pick an eigenvector $v_\lambda \in \mathscr{H}$ to each eigenvalue $\lambda \in \mathrm{Sp}(T)$ and consider the map
$$(x_i) \in \mathbb{C}^{\dim} \mapsto \sum_{i=1}^{n} x_i \lambda^i v_\lambda \in \mathscr{H}.$$
The computation of the Vandermonde determinant shows that this map is an isomorphism. Every element in the image of this map is also an element in Av. Thus $Av = \mathscr{H}$.

(b) Consider the map

$$\mathrm{ev}_T\colon \chi \in \widehat{A} \mapsto \chi(T) \in \mathbb{C}.$$

Let $\chi \in \widehat{A}$. We have

$$\chi(T - \chi(T)) = \chi(T) - \chi(T)\chi(1)) = 0.$$

Suppose $\mathrm{ev}_T(\chi) = \chi(T) \notin \mathrm{Sp}(T)$. All eigenvalues of $T - \chi(T)$ are non-zero, so the characteristic polynomial $\mathrm{char}_{T-\chi(T)}(X)$ is coprime to X. The Chinese remainder theorem says that there are polynomials $P, Q \in \mathbb{C}[X]$ with

$$1 = P\mathrm{char}_{T-\chi(T)}(X) + QX.$$

If we put $X = T - \chi(T)T$ in this relation, then the Cayley-Hamilton theorem says

$$1 = Q(T - \chi(T))(T - \chi(T)).$$

Thus $(T - \chi(T))^{-1} = Q(T - \chi(T)) \in A$. This implies

$$1 = \chi(1) = \chi((T - \lambda)^{-1}(T - \lambda)) = \chi(T - \chi(T))\chi((T - \chi(T))^{-1}) = 0.$$

This is a contradiction thus $\chi(T) \in \mathrm{Sp}(T)$.
Suppose $\lambda \in \mathrm{Sp}(T)$. Then there is $v \in \mathscr{H} - \{0\}$ with $Tv = \lambda v$. This implies that for each $x \in A$ there is a unique $\chi(x) \in \mathbb{C}$ with

$$xv = \chi(x)v.$$

The map $\chi\colon A \to \mathbb{C}$ is a character because

$$(xy)v = x(yv) = x(\chi(y)v) = \chi(y)(xv) = \chi(x)\chi(y)v.$$

It satisfies $\chi(T) = \lambda$. Thus the map ev_T defines a bijection between $\widehat{A}$ and $\mathrm{Sp}(T)$. Both sides are discrete spaces, so the map ev_T is a homeomorphism.

(c) Define the set $E_\lambda := \{v \in \mathscr{H} : Tv = \lambda v\}$ and the operator $P_\lambda := E(\{\lambda\})$ for each $\lambda \in \mathrm{Sp}(T)$. Note that we have

$$\int f(\lambda) dE_{x,y}(\lambda) = \sum_{\lambda \in \mathrm{Sp}(T)} f(\lambda) E_{x,y}(\lambda) = \sum_{\lambda \in \mathrm{Sp}(T)} f(\lambda) \langle P_\lambda x, y \rangle.$$

for all $x, y \in \mathscr{H}$. This implies

$$\langle Tx, y \rangle = \sum_{\lambda \in \mathrm{Sp}(T)} \lambda \langle P_\lambda x, y \rangle = \left\langle \sum_{i=1}^{n} \lambda P_\lambda x, y \right\rangle.$$

Thus

$$T = \sum_{\lambda \in \mathrm{Sp}(T)} \lambda P_\lambda.$$

For each $v \in \mathscr{H}$ we have

$$TP_{\lambda_0}v = \sum_{\lambda \in \mathrm{Sp}(T)} \lambda P_\lambda P_{\lambda_0} v.$$

Note that

$$P_\lambda P_{\lambda'} = E(\{\lambda\} \cap \{\lambda'\}) = E(\emptyset) = 0$$

if $\lambda \neq \lambda'$. Thus we get

$$TP_{\lambda_0}v = \lambda P_{\lambda_0}^2 v = \lambda P_{\lambda_0} v$$

because P_λ is a projection. This implies that the image of P_λ is contained in E_λ. We also have

$$\sum_{\lambda \in \mathrm{Sp}(T)} P_\lambda = 1.$$

Thus the image P_λ must be E_λ because, if it were smaller, the sum of the images of the P_λ could not generate $\mathscr{H}$. This uniquely determines P_λ as the orthogonal projection to E_λ. This completely describes E because

$$E(\{\lambda_1, \ldots, \lambda_n\}) = \sum_{i=1}^{n} P_{\lambda_i}$$

for all pairwise distinct $\lambda_i \in \mathrm{Sp}(T)$.

2. Let $\mathscr{H}$ be a Hilbert space and $U \in \mathscr{L}(\mathscr{H})$ a unitary operator. Define the C*-algebra

$$A := \overline{\{p(U, U^*) : p \in \mathbb{C}[X, Y]\}}.$$

This algebra is commutative, so the spectral theorem implies that there is a resolution of the identity E with

$$U = \int_{\widehat{A}} U dE.$$

Define the set $X := \{\chi \in \widehat{A} : \chi(U) = 1\}$.

(a) Let $\chi \in \widehat{A}$. Prove $|\chi(U)| = 1$. Apply the formula for the geometric series to obtain

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi(U^n) = 1_X(\chi)$$

for each $\chi \in \widehat{A}$ where 1_X denotes the indicator function of X.

(b) Let $W := \{v \in \mathscr{H} : Uv = v\}$ and denote by $P \colon \mathscr{H} \to \mathscr{H}$ the orthogonal projection onto W. Prove

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i v = Pv$$

for each $v \in \mathscr{H}$.

Hint: Use Exercise (a) and dominated convergence to prove

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} U^i = \int_{\widehat{A}} 1_X dE = E(X).$$

This result is known as von Neumann's ergodic theorem. Applications of this theorem can be found in [Zi, Ch. 4.4].

Solution:

(a) We have

$$1 = \chi(1) = \chi(UU^*) = \chi(U)\chi(U^*) = |\chi(U)|^2.$$

Consider a complex number $z \in \mathbb{C}$ with $z \neq 1$. The geometric series says

$$\frac{1}{n}\sum_{i=0}^{n-1} z^i = \frac{1}{n}\frac{1-z^n}{1-z}.$$

If $|z| = 1$, then

$$\left|\frac{1}{n}\frac{1-z^n}{1-z}\right| \leqslant \frac{2}{|1-z|n.}$$

This converges to 0 as $n \to \infty$. If $z = 1$, then the sum on the left evaluates to

$$\frac{1}{n}\sum_{i=0}^{n-1} z^n = 1.$$

(b) For each $n \geqslant 1$ define the operator

$$S_n := \frac{1}{n}\sum_{i=0}^{n-1} U^i - E(X).$$

Let $v \in \mathscr{H}$. We have

$$\langle S_n v, S_n v\rangle = \langle S_n^* S_n v, v\rangle.$$

Note that we have

$$\left|\frac{1}{n}\sum_{i=0}^{n-1}(\chi(U)^n - 1_X(\chi))\right|^2 \leqslant \left(\frac{1}{n}\sum_{i=0}^{n-1}(|\chi(U)^n| + 1)\right)^2 = 4.$$

Therefore, the first claim, exercise (a) and the dominated convergence theorem implies

$$\lim_{n\to\infty}\langle S_n^* S_n v, v\rangle = \lim_{n\to\infty}\int \left|\frac{1}{n}\sum_{i=0}^{n-1}(\chi(U)^n - 1_X)\right|^2 dE_{v,v} = 0$$

for each $v \in \mathscr{H}$. We can plug this into the previous equation to get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i v = E(X)v.$$

Let $v \in \mathscr{H}$. Note that we have

$$\frac{1}{n} \sum_{i=0}^{n-1} U^i v - \frac{1}{n} \sum_{i=1}^{n} U^i v = \frac{v + U^n v}{n}.$$

This converges to zero as $n \to \infty$, so we get

$$UE(X)v = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^{i+1} v = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i v = E(X)v.$$

Thus $\operatorname{im}(E(X)) \subset W$. Suppose $v \in W$, then

$$E(X)v = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} U^i v = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} v = v$$

because $U^i v = v$ for each $i \geqslant 0$ (since $v \in W$). Thus $\operatorname{im}(E(X)) = W$. Hence $E(X)$ is the orthogonal projection onto W.