Exercise Sheet 7

1. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathscr{H})$ a normal operator. Show that

$$\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| \leqslant 1\}$$

Hint: Use

$$\|T\| = \max\{|\lambda| : \lambda \in \mathrm{Sp}(T)\}$$

and the spectral theorem for normal operators.

2. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathscr{H})$ a normal operator. Suppose there is an operator $S \in \mathcal{B}(\mathscr{H})$ such that

$$S = \int_{\mathrm{Sp}(T)} f dE_T$$

for some $f \in \mathcal{B}^\infty(\mathrm{Sp}(T))$. This implies that S is a normal operator. Prove that the resolutions of the identity E_T and E_S associated with T and S by the spectral theorem are related by

$$E_S(\omega) = E_T(f^{-1}(\omega))$$

for each Borel set $\omega \subset \mathrm{Sp}(S)$.

3. Let G be a finite group and $H < G$ a proper subgroup. Recall that G acts on a set X doubly transitively if for all $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$ there is $g \in G$ with $x_1 = gx_2$ and $y_1 = gy_2$. Define the Hilbert space

$$\mathscr{H} := \left\{ f \in \ell^2(G/H) : \sum_{g \in G/H} f(gH) = 0 \right\}$$

and put $(\pi(g)f)(xH) := f(g^{-1}xH)$ for all $f \in \mathscr{H}$ and $g, x \in G$. This G-action defines a unitary representation of G on $\mathscr{H}$. Show that π is irreducible if and only if the G-action on G/H is doubly transitive.