Exercise Sheet 7 - Solutions

1. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathscr{H})$ a normal operator. Show that

$$\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| \leqslant 1\}$$

Hint: Use

$$\|T\| = \max\{|\lambda| : \lambda \in \mathrm{Sp}(T)\}$$

and the spectral theorem for normal operators.

Solution: Because T is normal, we have

$$\|T\| = \|T\|_{\mathrm{sp}}$$

where the right-hand side refers to the spectral norm of T in $\mathscr{L}(\mathscr{H})$. This says

$$\|T\| = \max\{|\lambda| : \lambda \in \mathrm{Sp}(T)\}.$$

Let $\lambda \in \mathrm{Sp}_{\mathscr{L}(\mathscr{H})}(T)$. I claim that there is a sequence $x_n \in \mathscr{H}$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty}(Tx_n - \lambda x_n) = 0$. Indeed, define $D_n := \{z \in \mathrm{Sp}(T) : |z - \lambda| < 1/n\}$ for each $n \geqslant 1$. Let E be the resolution of the identity on $\mathrm{Sp}(T)$ defined by the spectral theorem. Then $E(D_n) \neq 0$ by Theorem 5.20 (iv) in the lecture, so there exists $x_n \in \mathrm{im}(E(D_n))$ with $\|x_n\| = 1$. We have

$$\begin{aligned}\|Tx_n - \lambda x_n\|^2 &= \langle (T-\lambda)(T-\lambda)^* x_n, x_n\rangle \\ &= \int_{D_n} |T-\lambda|^2 dE_{x_n,x_n} \\ &\leqslant n^{-2}\int_{D_n} dE_{x_n,x_n} \leqslant n^{-2}.\end{aligned}$$

Now this implies

$$\|\lambda\| = \lim_{n\to\infty} |\langle Tx_n, x_n\rangle|$$

and hence

$$\|\lambda\| \leqslant \sup\{|\langle Tx, x\rangle| : \|x\| \leqslant 1\}.$$

Taking the supremum over all eigenvalues yields

$$\|T\| \leqslant \sup\{|\langle Tx, x\rangle| : \|x\| \leqslant 1\}.$$

Note that we have

$$|\langle Tx, x\rangle\| \leqslant \|Tx\|\|x\| \leqslant \|T\|\|x\|^2$$

for all $x \in \mathscr{H}$ by the Cauchy-Schwarz inequality. Hence

$$\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| \leqslant 1\}.$$

2. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathscr{H})$ a normal operator. Suppose there is an operator $S \in \mathcal{B}(\mathscr{H})$ such that
$$S = \int_{\mathrm{Sp}(T)} f dE_T$$
for some $f \in \mathcal{B}^\infty(\mathrm{Sp}(T))$. This implies that S is a normal operator. Prove that the resolutions of the identity E_T and E_S associated with T and S by the spectral theorem are related by
$$E_S(\omega) = E_T(f^{-1}(\omega))$$
for each Borel set $\omega \subset \mathrm{Sp}(S)$.

Solution: For each Borel set $\omega \subset \mathrm{Sp}(S)$ define $E(\omega) := E_T(f^{-1}(\omega))$. This is a resolution of the identity for $\mathrm{Sp}(S)$. For each $g \in \mathcal{B}^\infty(\mathrm{Sp}(S))$, an approximation argument with step-functions proves
$$\int_{\mathrm{Sp}(S)} g dE = \int_{\mathrm{Sp}(T)} g \circ f dE_T.$$
Define
$$B := \left\{ g \in C(\mathrm{Sp}(S)) : \int_{\mathrm{Sp}(S)} g dE_S = \int_{\mathrm{Sp}(S)} g dE \right\}.$$
This is a C*-algebra because the multiplication of two functions integrates to the composition of the operators, the complex conjugate of a function integrates to the adjoint operator, and the uniform limit of functions integrates to a uniform limit of operators. Note that $1 \in B$ because
$$\int_{\mathrm{Sp}(S)} dE_S = E_S(\mathrm{Sp}(S)) = \mathrm{id}_{\mathscr{H}} = E(\mathrm{Sp}(S)) = \int_{\mathrm{Sp}(S)} dE.$$
We also have
$$\int_{\mathrm{Sp}(S)} S dE = \int_{\mathrm{Sp}(T)} f dE_T = S = \int_{\mathrm{Sp}(S)} S dE_S.$$
Thus we get $S \in B$. Note that for each distinct $\lambda, \nu \in \mathrm{Sp}(S)$, we have $S(\lambda) \neq S(\nu)$ (because $S(\lambda) = \lambda$) and $1(\lambda) = 1$. Thus the assumptions of the Stone-Weierstrass theorem are satisfied and B is dense in $C(\mathrm{Sp}(S))$. Note that B is a complete subspace of $C(\mathrm{Sp}(S))$, so it is closed and hence $B = C(\mathrm{Sp}(S))$. In other words, each $T \in A$ satisfies
$$\int_{\mathrm{Sp}(S)} \widehat{T} dE = \int_{\mathrm{Sp}(S)} \widehat{T} dE_S.$$
The spectral theorem says that E_S is the unique resolution of the identity with this property. Thus $E = E_S$ and hence
$$E_S(\omega) = E(\omega) = E_T(f^{-1}(\omega))$$
for each Borel set $\omega \subset \mathrm{Sp}(S)$.

3. Let G be a finite group and $H < G$ a proper subgroup. Recall that G acts on a set X doubly transitively if for all $x_1, x_2, y_1, y_2 \in X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$ there is $g \in G$ with $x_1 = gx_2$ and $y_1 = gy_2$. Define the Hilbert space

$$\mathscr{H} := \left\{ f \in \ell^2(G/H) : \sum_{g \in G/H} f(gH) = 0 \right\}$$

and put $(\pi(g)f)(xH) := f(g^{-1}xH)$ for all $f \in \mathscr{H}$ and $g, x \in G$. This G-action defines a unitary representation of G on $\mathscr{H}$. Show that π is irreducible if and only if the G-action on G/H is doubly transitive.

Solution: Consider a G-equivariant map $T\colon \ell^2(G/H) \to \ell^2(G/H)$. There exists a kernel $K \in \ell^2(G/H \times G/H)$ such that

$$(Tf)(x) = \sum_{g \in G/H} f(g)K(g, x)$$

for all $f \in \ell^2(G/H)$ and $x \in G$. Each $g \in G$ satisfies

$$\pi(g)Tf(x) = \sum_{g' \in G/H} f(g')K(g', g^{-1}x)$$

and

$$T(\pi(g)f)(x) = \sum_{g' \in G/H} f(g^{-1}g')K(g', x) = \sum_{g' \in G/H} f(g')K(gg', x).$$

The G-equivariance of T implies

$$K(gg', x) = K(g', g^{-1}x).$$

Define the function $\phi\colon H\backslash G/H \to \mathbb{C}$ by $\phi(g) := K(1, g)/|H|$. We have

$$Tf(x) = \sum_{g \in G/H} f(g)K(g, x) = \sum_{g \in G/H} f(g)K(1, g^{-1}x) = \sum_{gg'=x} f(g)\phi(g') = f * \phi(x)$$

where $*$ denotes the convolution.

Conversely, any function $\phi\colon H\backslash G/H \to \mathbb{C}$ defines a G-equivariant map $T_\phi(f) := f * \phi$. This constructs a linear map

$$\Phi\colon \ell^2(H\backslash G/H) \to \mathrm{End}_G(\ell^2(G/H)),\ \phi \mapsto T_\phi.$$

where $\mathrm{End}_G(\ell^2(G/H))$ is the space of G-equivariant endomorphisms. Note that this map is injective because $T_\phi(\delta_1) = \phi$, where δ_1 is the function supported at the point $1 \in G/H$. Moreover, it is surjective by the above argument.

We have a map of sets

$$\varphi\colon H\backslash G/H \to G\backslash(G/H \times G/H),\ g \mapsto (1, g).$$

Suppose $\varphi(g) = \varphi(g')$. Then there is $g'' \in G$ such that

$$(1, g) = (g'', g''g').$$

The equation in the first coordinate says $g'' \in H$. The equation in the second coordinate implies $g \in Hg'H$. Hence the map is injective. Let $(g, g') \in G\backslash(G/H \times G/H)$. We have

$$(g, g') = (1, g^{-1}g') = \varphi(g^{-1}g'),$$

so φ is bijective.

Let $1 \in \ell^2(G/H)$ be the constant function. The trivial representation G-equivariantly injects into $\ell^2(G/H)$ via the map

$$\lambda \mapsto \lambda/|G/H|.$$

This is a section to the G-equivariant summation map

$$\ell^2(G/H) \to \mathbb{C},\ f \mapsto \sum_{g \in G/H} f(g)$$

whose kernel is π. Thus we get a G-equivariant splitting

$$\ell^2(G/H) \cong \pi \oplus \mathbb{C}.$$

Suppose G acts doubly transitively on G/H. Then

$$G\backslash(G/H \times G/H) \cong \{0, 1\}$$

consists of precisely two points, namely the orbit of the diagonal and the off-diagonal. This implies

$$\dim_{\mathbb{C}}(\mathrm{End}_G(\ell^2(G/H))) = 2$$

because Φ is an isomorphism. This implies

$$\dim_{\mathbb{C}}(\mathrm{End}_G(\pi)) = 1.$$

because $H < G$ is a proper subgroup. Thus $\mathscr{H}$ is irreducible.

Suppose π is irreducible. We consider two cases. Suppose $|G/H| > 2$. We have $\dim(\mathscr{H}) = |G/H| - 1 > 1$. Because $\mathscr{H}$ is irreducible, this implies that $\mathscr{H}$ is not the trivial representation. Schur's Lemma implies

$$\dim_{\mathbb{C}}(\mathrm{End}_G(\ell^2(G/H))) = 2.$$

Thus $|G\backslash(G/H \times G/H)| = 2$. This means there is only one off-diagonal orbit in $G/H \times G/H$ but this is precisely what it means for G to act doubly transitively. Suppose $|G/H| = 2$. Let $g \in G - H$. The double transitivity says that $(1, g)$ can be mapped to $(1, g)$ or $(g, 1)$ and this follows from $g(1, g) = (g, 1)$..