Exercise Sheet 8

1. Recall that a compact Hausdorff space S is called *totally disconnected* if the connected components of S are precisely the points. For each compact Hausdorff space S, we define

$$C^{\text{fin}}(S) := \{f \in C(S) : |\text{im}(f)| < \infty\}.$$

Note that $C^{\text{fin}}(S)$ is an involutive and unital subalgebra of $C(S)$. Let A be a Banach algebra, then define

$$A^{\text{fin}} = \{x \in A | \exists p \in \mathbb{C}[X] : p(x) = 0\}.$$

Note that $C^{\text{fin}}(S) = (C(S))^{\text{fin}}$.

(a) Prove that the following conditions are equivalent for a compact Hausdorff space S:

i. The space S is totally disconnected.

ii. For each pair of disctinct points $s_1, s_2 \in S$ there exists a clopen decomposition $S = S_1 \sqcup S_2$ such that $s_1 \in S_1$ and $s_2 \in S_2$.

iii. For each pair of distinct points $s_1, s_2 \in S$ there exists a continuous function $\phi\colon S \to \{0, 1\}$ such that $\phi(s_1) = 0$ and $\phi(s_2) = 1$.

iv. The subalgebra $C^{\text{fin}}(S) \subset C(S)$ is dense.

Hint: To prove i. implies ii, use that the connected component $Q \subset S$ of a point $x \in S$ in a compact Hausdorff space S can be written as

$$Q = \bigcap_{\substack{x \in C \subset S \\ C \text{ clopen}}} C$$

Prove iii. implies iv. by using the Stone-Weierstrass theorem.

(b) Let X be a compact Hausdorff space. Prove that there exists a totally disconnected compact Hausdorff space S and a continuous surjection $S \to X$.

Hint: Consider the inclusion

$$C(X) \to C^b(X^{\text{disc}}) = \{f \in \mathbb{C}^X | \exists C \geqslant 0 \ \forall x \in X : |f(x)| \leqslant C\}$$

where $C^b(X^{\text{disc}})$ is the set of bounded continuous functions on the discrete space X^{disc} i.e. the set of all bounded functions $X \to \mathbb{C}$. The algebra $C^b(X^{\text{disc}})$ is a C*-algebra with

$$(C^b(X^{\text{disc}}))^{\text{fin}} = \{f \in \mathbb{C}^X || \text{im}(f)| < \infty\}.$$

Prove that $(C^b(X^{\text{disc}}))^{\text{fin}} \subset C^b(X^{\text{disc}})$ is a dense subspace. Then the induced morphism on the Guelfand spectrum

$$\widehat{C^b(X^{\text{disc}})} \to X$$

is a surjection from a totally disconnected compact Hausdorff space to X.

2. Prove that the quotient groups $\mathbb{Q}_p/\mathbb{Z}_p$ are discrete for each prime $p \in \mathbb{N}$. Moreover, prove that they are isomorphic to
$$\{z \in \mathbb{T} | \exists n \geqslant 1 : z^{p^n} = 1\}$$
as abstract groups.

3. Let G be a connected topological group and $\Gamma \lhd G$ a discrete normal subgroup. Show that Γ is contained in the center of G.