Exercise Sheet 8 - Solutions

1. Recall that a compact Hausdorff space S is called *totally disconnected* if the connected components of S are precisely the points. For each compact Hausdorff space S, we define

$$C^{\mathrm{fin}}(S) := \{f \in C(S) : |\mathrm{im}(f)| < \infty\}.$$

Note that $C^{\mathrm{fin}}(S)$ is an involutive and unital subalgebra of $C(S)$. Let A be a Banach algebra, then define

$$A^{\mathrm{fin}} = \{x \in A | \exists p \in \mathbb{C}[X] : p(x) = 0\}.$$

Note that $C^{\mathrm{fin}}(S) = (C(S))^{\mathrm{fin}}$.

(a) Prove that the following conditions are equivalent for a compact Hausdorff space S:

i. The space S is totally disconnected.

ii. For each pair of disctinct points $s_1, s_2 \in S$ there exists a clopen decomposition $S = S_1 \sqcup S_2$ such that $s_1 \in S_1$ and $s_2 \in S_2$.

iii. For each pair of distinct points $s_1, s_2 \in S$ there exists a continuous function $\phi\colon S \to \{0,1\}$ such that $\phi(s_1) = 0$ and $\phi(s_2) = 1$.

iv. The subalgebra $C^{\mathrm{fin}}(S) \subset C(S)$ is dense.

Hint: To prove i. implies ii, use that the connected component $Q \subset S$ of a point $x \in S$ in a compact Hausdorff space S can be written as

$$Q = \bigcap_{\substack{x \in C \subset S \\ C \text{ clopen}}} C$$

Prove iii. implies iv. by using the Stone-Weierstrass theorem.

(b) Let X be a compact Hausdorff space. Prove that there exists a totally disconnected compact Hausdorff space S and a continuous surjection $S \to X$.

Hint: Consider the inclusion

$$C(X) \to C^b(X^{\mathrm{disc}}) = \{f \in \mathbb{C}^X | \exists C \geqslant 0 \ \forall x \in X : |f(x)| \leqslant C\}$$

where $C^b(X^{\mathrm{disc}})$ is the set of bounded continuous functions on the discrete space X^{disc} i.e. the set of all bounded functions $X \to \mathbb{C}$. The algebra $C^b(X^{\mathrm{disc}})$ is a C*-algebra with

$$(C^b(X^{\mathrm{disc}}))^{\mathrm{fin}} = \{f \in \mathbb{C}^X || \mathrm{im}(f)| < \infty\}.$$

Prove that $(C^b(X^{\mathrm{disc}}))^{\mathrm{fin}} \subset C^b(X^{\mathrm{disc}})$ is a dense subspace. Then the induced morphism on the Guelfand spectrum

$$\widehat{C^b(X^{\mathrm{disc}})} \to X$$

is a surjection from a totally disconnected compact Hausdorff space to X.

Solution:

(a) Suppose S is totally disconnected and consider distinct $s_1, s_2 \in S$. The hint says that there is a clopen subset $s_1 \in S_1 \subset S$ such that $s_2 \notin S_1$. Thus we get $s_2 \in S_2 := S_1^c$.

Suppose condition ii. is satisfied and consider distinct $s_1, s_2 \in S$. Then there exists a clopen $S_1 \subset S$ with $s_1 \in S_1$ and $s_2 \notin S_1$. The indicator function $\phi := 1_{S_1^c}$ is continuous because S_1 is both open and closed. This function satisfies $\phi(s_1) = 0$ and $\phi(s_2) = 1$.

Suppose condition iii. is satisfied. Note that the subalgebra $C^{\text{fin}}(S) \subset C(S)$ is involutive and unital, so the conditions of the Stone-Weierstrass theorem are satisfied if the algebra can separate points. The condition iii. implies that functions with finite image separate points. Thus condition iv. is satisfied by the Stone-Weierstrass theorem.

Suppose condition iv. is satisfied. Consider two distinct points $s_1, s_2 \in S$. Urysohns Lemma implies that there is a continuous function $\phi \in C(S)$ such that $\phi(s_1) = 0$ and $\phi(s_2) = 1$. There exists an approximation $\tilde{\phi} \in C^{\text{fin}}(S)$ such that $\|\phi - \tilde{\phi}\| < 1/2$. Let $D := \{z \in \mathbb{C} : |z| < 1/2\}$ and $\tilde{D} := \{z \in D : z \in \operatorname{im}(\tilde{\phi})\}$. Note that

$$\tilde{\phi}^{-1}(D) = \tilde{\phi}^{-1}(\tilde{D})$$

is an equality of a closed set and an open set. Thus $\phi^{-1}(D)$ is open and closed. Moreover, $s_1 \in \phi^{-1}(D)$ and $s_2 \notin \phi^{-1}(D)$, so the condition ii. is satisfied.

Suppose condition ii. is satisfied. Consider a connected component Q of a point $x \in S$. Suppose there is $y \in Q$ such that $y \neq x$. Then there is a clopen $S_1 \subset S$ such that $x \in S_1$ and $y \notin S_1$. Thus $Q \cap S_1 = S_1$ but this contradicts $y \in Q$. Thus $y \notin Q$ and hence $Q = \{x\}$.

(b) Define the algebra $A := C^b(X^{\text{disc}}) := \{f \in \mathbb{C}^X | \exists C \geqslant 0 \ \forall x \in X : |f(x)| \leqslant C\}$. It is a C*-algebra.

Let $f \in A$ and consider a polynomial $p = \prod_i (X - \alpha_i)$, $\alpha_i \in \mathbb{C}$. Then $p(f)(x) = \prod_{i=1}^n (f(x) - \alpha_i)$ so $p(f) = 0$ is equivalent to $\operatorname{im}(f) \subset \{\alpha_i : 1 \leqslant i \leqslant n\}$. This implies

$$(C^b(X^{\text{disc}}))^{\text{fin}} = \{f \in \mathbb{C}^X : |\operatorname{im}(f)| < \infty\}.$$

Consider any function $f \in A$ and $\epsilon > 0$. Cover the image of f by a finite, disjoint union $D_1, \ldots, D_n \subset \mathbb{C}$ of non-empty subsets each which are contained in balls of radius $< \epsilon$ centered around a point $x_i \in D_i$. Define

$$\tilde{f}(x) := \sum_{i=1}^n x_i 1_{f^{-1}(D_i)}$$

Note that $\tilde{f} \in A^{\text{fin}}$ and $\|f - \tilde{f}\| < \epsilon$, thus we get that $A^{\text{fin}} \subset A$ is a dense subalgebra. The Guelfand isomorphism provides an identification $A^{\text{fin}} \cong C(\widehat{A})^{\text{fin}}$ of subspaces in $A \cong C(\widehat{A})$. Since $A^{\text{fin}} \subset A$ is dense, this implies $C(\widehat{A})^{\text{fin}} = C^{\text{fin}}(\widehat{A}) \subset C(\widehat{A})$ is dense. Thus Exercise (a) implies that $\widehat{A}$ is totally disconnected.

Suppose the complement of the image of

$$\phi \colon \widehat{A} \to X$$

is non-empty. The space $\widehat{A}$ is compact, so by continuity the complement of $\phi(\widehat{A})$ is open. Urysohns's Lemma implies that there is a function $\psi \in C(X)$ such that $\psi \neq 0$ and $\psi \circ \phi = 0$. But this contradicts the injectivity of $C(X) \to A$. Thus

$$\widehat{A} \to X$$

is surjective.

2. Prove that the quotient groups $\mathbb{Q}_p/\mathbb{Z}_p$ are discrete for each prime $p \in \mathbb{N}$. Moreover, prove that they are isomorphic to

$$\{z \in \mathbb{T} | \exists n \geqslant 1 : z^{p^n} = 1\}$$

as abstract groups.

Solution: To prove that $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete, means to prove that $\{x + \mathbb{Z}_p\} \subset \mathbb{Q}_p/\mathbb{Z}_p$ is open for each $x \in \mathbb{Q}_p$. Note that the preimage under the quotient map $\pi\colon \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$ is given by $\pi^{-1}(x + \mathbb{Z}_p) = x + \mathbb{Z}_p$. Because $\mathbb{Z}_p$ is open we get that $x + \mathbb{Z}_p$ is open. Since π is a quotient map, this implies the point $\{x + \mathbb{Z}_p\}$ is open. Thus the quotient is discrete.

We construct the map as follows: each non-zero $x \in \mathbb{Q}_p$ has a unique power series expansion

$$x = \sum_{n \geqslant N} x_n p^n$$

where $x_n \in \{0, \dots, p-1\}$ and $N \in \mathbb{Z}$ such that $a_N \neq 0$. Define $\chi\colon \mathbb{Q}_p \to \{z \in \mathbb{T} | \exists n \geqslant 1 : z^{p^n} = 1\}$ by the formula

$$\chi(x) := \exp\left(\sum_{0 > n \geqslant N} 2\pi i x_n p^n \right)$$

Note that $\chi(x + y) = \chi(x)\chi(y)$. The kernel of the map is $\mathbb{Z}_p$ and the map is surjective, so χ factors through to an isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \{z \in \mathbb{T} | \exists n \geqslant 1 : z^{p^n} = 1\}.$$

Note, however, that χ does not induce an isomorphism of topological groups. This morphism is only continuous but not open.

3. Let G be a connected topological group and $\Gamma \lhd G$ a discrete normal subgroup. Show that Γ is contained in the center of G.

Solution: Let $\gamma \in \Gamma$ and consider the conjugation map

$$c\colon G \to G, \; g \mapsto g\gamma g^{-1}.$$

The subgroup Γ is normal, so the image of c is contained in Γ. Thus the conjugation map factors as

$$\tilde{c}\colon G \to \Gamma$$

with $i \circ \tilde{c} = c$ where $i\colon \Gamma \to G$ is the inclusion. The map $\tilde{c}$ is a continuous map from a connected space to a discrete space hence it is constant. It satisfies $c(1) = \gamma$, so

$$g\gamma g^{-1} = c(g) = i(\tilde{c}(g)) = i(\tilde{c}(1)) = c(1) = \gamma$$

for all $g \in G$. Thus $\gamma \in Z(G)$.