Exercise Sheet 9

1. Let F be a finite abelian group. Prove $\widehat{F} \cong F$.

2. For each $a \in \mathbb{R}$, define the additive character $\chi_a(t) := e^{2\pi i a t}$ for $t \in \mathbb{R}$. Prove that the map
$$\Psi \colon \mathbb{R} \to \widehat{\mathbb{R}},\ a \mapsto \chi_a$$
is an isomorphism of topological groups.

3. Deduce $\widehat{\mathbb{T}} \cong \mathbb{Z}$ from Exercise 2.

4. Let F_n be a finite abelian group for all $n \geqslant 1$ and put $G := \prod_{i \geqslant 1} F_n$ with the product topology. Show that
$$\widehat{G} \cong \bigoplus_{n \geqslant 1} F_n$$
as topological groups where the group on the right is equipped with the discrete topology.
Hint: Prove that there is a neighborhood $1 \in V \subset \mathbb{T}$ of the identity such that any subgroup $H < \mathbb{T}$ with $H \subset V$ satisfies $H = \{1\}$.

5. Let $\chi \colon \mathbb{Q}_p \to \mathbb{T}$ be the continuous character constructed in the lecture (with $\ker(\chi) = \mathbb{Z}_p$). For each $a \in \mathbb{Q}_p$, define $\chi_a(t) := \chi(at)$ for each $t \in \mathbb{Q}_p$. Show that the map
$$\Psi \colon \mathbb{Q}_p \to \widehat{\mathbb{Q}_p},\ a \mapsto \chi_a$$
is an isomorphism of topological groups.
Hint: Prove that for each character $\gamma \in \widehat{G}$ there is $n \in \mathbb{Z}$ such that $\gamma(p^n x) = 1$ for all $x \in \mathbb{Z}_p$. Thus $\tilde{\gamma}(x) := \gamma(p^n x)$ factors through $\mathbb{Q}_p/\mathbb{Z}_p$. Define the subgroups
$$A_{-n} := \{x \in \mathbb{Q}_p/\mathbb{Z}_p : p^n x = 0\}$$
for $n \geqslant 1$. Determine the group $\widehat{\mathbb{Q}_p/\mathbb{Z}_p}$ by noting that the diagram
$$\cdots \hookrightarrow A_{-(n-1)} \hookrightarrow A_{-n} \hookrightarrow A_{-(n+1)} \hookrightarrow \cdots$$
dualizes to
$$\cdots \twoheadleftarrow \widehat{A}_{-(n-1)} \twoheadleftarrow \widehat{A}_{-n} \twoheadleftarrow \widehat{A}_{-(n+1)} \hookrightarrow \cdots$$
under the Pontryagin dual.

6. Let $K_j < G_j$ be a compact subgroup in a LCH group G_j indexed by $j \in J$. Show that the restricted product $\prod'_{j \in J} G_j$ is a locally compact Hausdorff space.

7. Define the adeles
$$\mathbb{A}_{\mathbb{Q}} := \mathbb{R} \times \prod_p{}' \mathbb{Q}_p$$
to be the product of $\mathbb{R}$ with the restricted restricted product over all primes $p \in \mathbb{N}$, where we take the product with respect to the subgroups $\mathbb{Z}_p \subset \mathbb{Q}_p$.

(a) Prove that the multiplication map $\mathbb{A}_\mathbb{Q} \times \mathbb{A}_\mathbb{Q} \to \mathbb{A}_\mathbb{Q},\ (x, y) \mapsto xy$ is continuous.

(b) Consider the diagonal map $\Delta\colon \mathbb{Q} \to \mathbb{A}_\mathbb{Q},\ x \mapsto (x, x)$. Prove that the image of Δ is a discrete, closed subgroup of $\mathbb{A}_\mathbb{Q}$.

(c) Prove that the quotient $\mathbb{A}_\mathbb{Q}/\Delta(\mathbb{Q})$ is compact.

8. Show that for every $x \in \mathbb{Q}_p$ there exist unique $a_n \in \{0, \ldots, p-1\}$ for $n \geqslant v_p(x)$ such that

$$x = \sum_{n \geqslant v_p(x)} a_n p^n.$$

9. Using Exercise 8, we define the fractional part of a p-adic number $x \in \mathbb{Q}_p$ to be

$$\{x\} := \sum_{0 > n > v_p(x)} a_n p^n \in \mathbb{Q}.$$

where $x = \sum_{n \geqslant v_p(x)} a_n p^n$. Define the map $\chi\colon \mathbb{A}_\mathbb{Q} \to \mathbb{T}$ by

$$\chi(x_\infty, x_p) := e^{2\pi i \{x_\infty\}} \prod_p e^{2\pi i \{x_p\}}.$$

(a) Prove that χ is a continuous character on $\mathbb{A}_\mathbb{Q}$.

(b) Define the character $\chi_a(x) := \chi(ax)$ for all $a, x \in \mathbb{A}_\mathbb{Q}$. Show that the map

$$\mathbb{A}_\mathbb{Q} \to \widehat{\mathbb{A}_\mathbb{Q}},\ a \mapsto \chi_a$$

is an isomorphism of topological groups.