Exercise Sheet 9 - Solutions

1. Let F be a finite abelian group. Prove $\widehat{F} \cong F$.

   Solution: Let G_1, G_2 be topological groups. Every character $\chi\colon G_1 \times G_2 \to \mathbb{T}$ satisfies
   $$\chi(g_1, g_2) = \chi(g_1, 1)\chi(1, g_2)$$
   for all $g_1, g_2 \in G$. This means we get an isomorphism
   $$\widehat{G_1} \times \widehat{G_2} \to \widehat{G_1 \times G_2},\ (\chi_1, \chi_2) \mapsto ((g_1, g_2) \mapsto \chi_1(g_1)\chi_2(g_2))$$
   with inverse
   $$\widehat{G_1 \times G_2} \to \widehat{G_1} \times \widehat{G_2},\ \chi \mapsto ((g_1 \mapsto \chi(g_1, 1)), (g_2 \mapsto \chi(1, g_2)).$$
   Any finite abelian group can be written as a product of cyclic groups thus we can assume F cyclic by the first step. In this case, $F \cong \widehat{F}$ is proven in Example 7.6 (ii).

2. For each $a \in \mathbb{R}$, define the additive character $\chi_a(t) := e^{2\pi i a t}$ for $t \in \mathbb{R}$. Prove that the map
   $$\Psi\colon \mathbb{R} \to \widehat{\mathbb{R}},\ a \mapsto \chi_a$$
   is an isomorphism of topological groups.

   Solution: We define
   $$\lambda \cdot \chi(x) := \chi(\lambda x)$$
   for all $\lambda \in \mathbb{R}$, $\chi \in \widehat{\mathbb{R}}$ and $x \in \mathbb{R}$. This action turns $\widehat{\mathbb{R}}$ into a real, topological vector space. Note that it is complete because it is locally compact. Applying the local compactness again implies that $\widehat{\mathbb{R}}$ is finite-dimensional.

   Consider a non-trivial character $\chi \in \widehat{\mathbb{R}}$. Because $\mathbb{R}$ is simply-connected, there exists a unique, continuous logarithm
   $$\psi\colon \mathbb{R} \to \mathbb{R}$$
   such that $\psi(0) = 0$ and $e^{2\pi i \psi(x)} = \chi(x)$. For each $y \in \mathbb{R}$, note that
   $$f(x) := \psi(x) + \psi(y)$$
   is a continuous logarithm of $x \mapsto \chi(x + y)$ with $f(0) = \psi(x)$. This determines $f(x)$ uniquely and hence
   $$\psi(x + y) = f(x) = \psi(x) + \psi(y)$$
   for all $x \in \mathbb{R}$. Thus ψ is additive and continuous. This implies ψ is $\mathbb{R}$-linear. Hence there is $a \in \mathbb{R}$ such that
   $$\psi(x) = ax$$

for all $x \in \mathbb{R}$. Thus Ψ is surjective. Note that Ψ is injective because

$$\chi_a'(0) = 2\pi i a.$$

The map Ψ is $\mathbb{R}$-linear because

$$\Psi(ab)(x) = \chi_{ab}(x) = e^{2\pi i(ab)x} = e^{2\pi a(bx)} = \chi_a(bx) = (b \cdot \chi_a)(x).$$

Thus Ψ is an isomorphism of complete vector spaces. The Hahn-Banach theorem implies that is a homeomorphism.

We give a second proof. Let V be a finite-dimensional $\mathbb{R}$ vector space. Then $\widehat{V}$ is a finite-dimensional $\mathbb{R}$-vector space with the definition

$$\lambda\chi(v) := \chi(\lambda v)$$

for all $\lambda \in \mathbb{R}$ and $v \in V$. The Pontryagin duality map $D\colon V \to \widehat{\widehat{V}}$ is $\mathbb{R}$-linear because

$$D(av)(\chi) = \chi(av) = (a \cdot \chi)(v) = D(v)(a \cdot \chi) = (a \cdot D(v))(\chi).$$

Let $\widehat{\mathbb{R}} \cong \mathbb{R}^n$. The Pontryagin duality theorem implies that we have $\mathbb{R}$-linear isomorphisms

$$\mathbb{R} = \widehat{\widehat{\mathbb{R}}} \cong \mathbb{R}^{n^2}.$$

Thus $n^2 = 1$ and hence $n = 1$. Note that Ψ is $\mathbb{R}$-linear because

$$\Psi(ab)(x) = \chi_{ab}(x) = e^{2\pi i(ab)x} = e^{2\pi i a(bx)} = \chi_a(bx).$$

Furthermore, the map is non-zero, so it is an isomorphism.

3. Deduce $\widehat{\mathbb{T}} \cong \mathbb{Z}$ from Exercise 2.

   Solution: Let $H < G$ be a closed subgroup in a LCA group G. The mapping property of the quotient G/H says that a continuous group morphism $\varphi\colon G \to \mathbb{T}$ factors continuously through G/H if and only if $H \subset \ker(\varphi)$.

   We have $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, so we get a continuous bijection

$$\widehat{\mathbb{T}} \to \{\chi \in \widehat{\mathbb{R}} : \chi(1) = 1\} \cong \mathbb{Z}$$

   where the space on the right is equipped with the subspace topology. The space $\widehat{\mathbb{T}}$ is discrete because $\mathbb{T}$ is compact. Thus the above map is an isomorphism of topological groups.

4. Let F_n be a finite abelian group for all $n \geqslant 1$ and put $G := \prod_{i\geqslant 1} F_n$ with the product topology. Show that

$$\widehat{G} \cong \bigoplus_{n\geqslant 1} \widehat{F_n}$$

   as topological groups where the group on the right is equipped with the discrete topology.

Hint: Prove that there is a neighborhood $1 \in V \subset \mathbb{T}$ *of the identity such that any closed subgroup* $H < \mathbb{T}$ *with* $H \subset V$ *satisfies* $H = \{1\}$.

Solution: Let $x \in \mathbb{T}$ and consider the closed subgroup generated by x in $\mathbb{T}$. If x has finite order, then this subgroup is finite. Suppose x does not have finite order and let $\epsilon > 0$. By the pigeonhole principle, there are $n, m \in \mathbb{Z}$ such that $nx - mx = k + \epsilon'$ for some $k \in \mathbb{Z}$ and $0 < \epsilon' < \epsilon$. This implies $\epsilon' \in \mathbb{T}$. Therefore, H is dense in $\mathbb{T}$ and thus $H = \mathbb{T}$.

Therefore any closed subgroup in $\mathbb{T}$ is either finite or $\mathbb{T}$. Define $V := \{z \in \mathbb{T} : \mathrm{Re}(z) > 0\}$. Any closed subgroup $H \subset V$ is finite. Moreover, if H is not a trivial subgroup, it has an element of order $\geqslant 5$. But this is not possible, because the powers of any element of order $\geqslant 5$ have an element in each quadrant of the complex plane.

Let $\chi \colon G \to \mathbb{T}$ be a continuous character. Then $\chi^{-1}(V)$ is open and non-empty. This implies that there is a finite subset $J \subset \mathbb{N}$ with

$$W := \{(x_n) \in G | \forall j \in J : x_j = 1\} \subset \chi^{-1}(V).$$

The subset W is a compact subgroup of G, hence $\chi(W) \subset V$ is a closed subgroup. This implies $\chi(W) = 1$. Deinfe $\chi_j(x) := \chi(1, \dots, 1, x, 1, \dots)$ for all $j \in J$ and $x \in F_j$. Then

$$\chi((x_n)_n) = \prod_{j \in J} \chi_j(x_j).$$

This implies that the multiplication map

$$\bigoplus_{n \geqslant 1} \widehat{F_n} \to \widehat{G}$$

is bijective. Proposition 7.3 says that the group on the right is discrete, so this map is an isomorphism of topological groups.

Here is a second argument to prove this claim. Let $H \triangleleft G$ be a closed subgroup in a topological group G with finite index. Then

$$G - H = \bigcup_{\substack{g \in G/H \\ g \notin H}} gH$$

is a finite union of closed sets. Thus $G - H$ is closed in G which says that H is open.

Consider two distinct elements $(x_n), (y_n) \in G$. There exists $m \geqslant 1$ such that $x_m \neq y_m$. The subset

$$Z := \{(z_n) \in G : z_m = x_m\}$$

is compact, has compact complement, and satisfies $(x_n) \in Z$ and $(y_n) \notin Z$. In particular, $\chi(Z)$ is clopen. This implies that $\chi(G)$ is totally disconnected. Since $\chi(G)$ is a closed subgroup of $\mathbb{T}$, the image is finite (because $\mathbb{T}$ is connected the image cannot be $\mathbb{T}$). Define for each finite set $J \subset \mathbb{N}$ the set

$$W_J := \{(x_n) \in G | \forall j \in J : x_j = 1\}.$$

The W_J form a neighborhood basis for G at 1. Since the image of χ is finite, the kernel is an open subgroup because it has finite index. Thus there is a finite subset $J \subset \mathbb{N}$ such that $W_J \subset \ker(\chi)$. In other words, we get $\chi(x) = 1$ for all $x \in W_J$. The same argument as before proves that the map is an isomorphism.

5. Let $\chi \colon \mathbb{Q}_p \to \mathbb{T}$ be the continuous character constructed in the lecture (with $\ker(\chi) = \mathbb{Z}_p$). For each $a \in \mathbb{Q}_p$, define $\chi_a(t) := \chi(at)$ for each $t \in \mathbb{Q}_p$. Show that the map

$$\Psi \colon \mathbb{Q}_p \to \widehat{\mathbb{Q}_p},\ a \mapsto \chi_a$$

is an isomorphism of topological groups.

Hint: Prove that for each character $\gamma \in \widehat{G}$ there is $n \in \mathbb{Z}$ such that $\gamma(p^n x) = 1$ for all $x \in \mathbb{Z}_p$. Thus $\tilde{\gamma}(x) := \gamma(p^n x)$ factors through $\mathbb{Q}_p/\mathbb{Z}_p$. Define the subgroups

$$A_{-n} := \{x \in \mathbb{Q}_p/\mathbb{Z}_p : p^n x = 0\}$$

for $n \geqslant 1$. Determine the group $\widehat{\mathbb{Q}_p/\mathbb{Z}_p}$ by noting that the diagram

$$\cdots \hookrightarrow A_{-(n-1)} \hookrightarrow A_{-n} \hookrightarrow A_{-(n+1)} \hookrightarrow \cdots$$

dualizes to

$$\cdots \twoheadleftarrow \widehat{A}_{-(n-1)} \twoheadleftarrow \widehat{A}_{-n} \twoheadleftarrow \widehat{A}_{-(n+1)} \hookrightarrow \cdots$$

under the Pontryagin dual.

Solution: The group $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete, so any character is continuous. Denote by $i_n \colon A_{-n} \to A_{-(n+1)}$ the inclusion. Note that we have an isomorphism

$$A_{-n} \to \mathbb{Z}/p^n\mathbb{Z},\ a \mapsto p^n a.$$

Under this isomorphism, we have $i_n(a) = pa$. We have another isomorphism

$$\mathbb{Z}/p^n\mathbb{Z} \to \widehat{\mathbb{Z}/p^n\mathbb{Z}},\ a \mapsto (t \mapsto e^{2\pi i a t/p^n}).$$

Under this isomorphism, the dual of i_n is given by

$$\widehat{i}_n(a) = a$$

i.e. it is the reduction modulo p^n map. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is the union of the subgroups A_{-n}, the above isomorphisms compose to

$$\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \to \{(\chi_n) \in \prod_{n\geqslant 1} \widehat{A}_{-n} : \chi_{n+1} \circ i_n = \chi_n\} = \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p.$$

Since any character is continuous, this map is a bijection. The composition of this bijection with a projection is given by

$$\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \to \widehat{A_{-n}},\ \chi \mapsto \chi|_{A_{-n}}.$$

This map is continuous. Thus the above map is a bijection of compact Hausdorff spaces and hence a homomorphism.

For each $x \in \mathbb{Z}_p$ and $\chi \in \widehat{\mathbb{Q}_p/\mathbb{Z}_p}$ define $x \cdot \chi(a) := \chi(xa)$. Note that the morphism

$$\psi \colon \widehat{\mathbb{Q}_p/\mathbb{Z}_p} \to \mathbb{Z}_p$$

satisfies

$$x \cdot \psi(\chi) = \psi(x \cdot \chi).$$

Thus for every pair $\gamma_1, \gamma_2 \in \widehat{\mathbb{Q}_p/\mathbb{Z}_p}$ there is $x \in \mathbb{Z}_p$ such that $x \cdot \gamma_1 = \gamma_2$ or $\gamma_1 = x \cdot \gamma_2$.

For each $x \in \mathbb{Q}_p$ and $\chi \in \widehat{\mathbb{Q}_p}$ define $x \cdot \chi(a) := \chi(xa)$. Let $\gamma \in \widehat{\mathbb{Q}_p}$. The image of the restriction $\gamma|_{\mathbb{Z}_p}$ is compact and totally disconnected hence it is finite (see the second proof of the previous exercise). Thus the kernel of $\gamma|_{\mathbb{Z}_p}$ is an open subgroup of $\mathbb{Z}_p$. A neighborhood basis of $\mathbb{Z}_p$ is given by $p^n\mathbb{Z}_p$ for $n \geqslant 0$, thus there is $n \geqslant 0$ such that $p^n\mathbb{Z}_p \subset \ker(\gamma)$. This implies that $p^n \cdot \gamma$ factors through $\mathbb{Q}_p/\mathbb{Z}_p$.

The above implies that for every pair $\gamma_1, \gamma_2 \in \widehat{\mathbb{Q}_p}$ there is $x \in \mathbb{Q}_p$ such that $x \cdot \gamma_1 = \gamma_2$. The morphism Ψ is $\mathbb{Q}_p$-linear because $\chi_a = a \cdot \chi$ for all $a \in \mathbb{Q}_p$. The space $\mathbb{Q}_p$ is one-dimensional and Ψ is a surjection onto a non-zero vector space, so Ψ is a bijection.

Define $U := \{\chi \in \widehat{\mathbb{Q}_p} : \chi|_{\mathbb{Z}_p} = 1\}$. I claim that this is an open subgroup of $\widehat{\mathbb{Q}_p}$. The first step says that for every $\chi \in \widehat{\mathbb{Q}_p}$ there is $n \geqslant 1$ such that $p^n \cdot \chi \in U$. Thus

$$\widehat{\mathbb{Q}_p} = \bigcup_{n \geqslant 0} p^{-n} U.$$

The Baire category theorem implies that U has to contain an open subset $V \subset U$. This implies that U is an open subgroup because

$$U = \bigcup_{u \in U} uV.$$

We have

$$\Psi(\mathbb{Z}_p) = U$$

because the kernel of χ is $\mathbb{Z}_p$. Thus we get

$$\Psi(p^n\mathbb{Z}_p) = p^n U.$$

The subsets $p^n\mathbb{Z}_p$ form a neighborhood basis of the identity. Therefore Ψ is open and hence a homomorphism.

6. Let $K_j < G_j$ be a compact subgroup in a LCH group G_j indexed by $j \in J$. Show that the restricted product $\prod'_{j \in J} G_j$ is a locally compact Hausdorff space.

   Solution: Let $X_1, \ldots, X_n$ be locally compact spaces and consider $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$. For each $1 \leqslant i \leqslant n$, there is a compact neighborhood $x_i \in C_i \subset X_i$. Then

$$x \in C_1 \times \ldots \times C_n \subset X_1 \times \cdots \times X_n$$

is a compact neighbourhood of x. Thus $X_1 \times \cdots X_n$ is locally compact space.

Let $(x_j) \in \prod'_{j \in J} G_j$. There exists a finite subset $I \subset J$ such that $x_i \in K_j$ for all $j \notin I$. This implies

$$x \in \prod_{j \in J} H_j =: U$$

where $H_j = G_j$ if $j \in I$ and $H_j = K_j$ otherwise. By definition, U is an open subset of the restricted product. Moreover, Tchynoff's theorem implies that U is a finite product of locally comapct spaces. Thus the restricted product is LCH.

Consider distinct elements $(x_j), (y_j) \in \prod'_{j \in J} G_j$. There exists $i \in J$ such that $x_i \neq y_i$. Let $U_1, U_2 \subset G_i$ be two disjoint neighbourhoods of x_i and y_i respectively. Then the subsets

$$V_1 := \{(x_j) \in \prod_{j \in J}{}' G_j : x_i \in V_1\}$$
$$V_2 := \{(x_j) \in \prod_{j \in J}{}' G_j : x_i \in V_2\}$$

are disjoint open neighbourhoods of x and y in the restricted product. Thus the restricted product is Hausdorff.

7. Define the adeles

$$\mathbb{A}_{\mathbb{Q}} := \mathbb{R} \times \prod_p{}' \mathbb{Q}_p$$

to be the product of $\mathbb{R}$ with the restricted restricted product over all primes $p \in \mathbb{N}$, where we take the product with respect to the subgroups $\mathbb{Z}_p \subset \mathbb{Q}_p$.

(a) Prove that the multiplication map $\mathbb{A}_{\mathbb{Q}} \times \mathbb{A}_{\mathbb{Q}} \to \mathbb{A}_{\mathbb{Q}},\ (x, y) \mapsto xy$ is continuous.

(b) Consider the diagonal map $\Delta\colon \mathbb{Q} \to \mathbb{A}_{\mathbb{Q}},\ x \mapsto (x, x)$. Prove that the image of Δ is a discrete, closed subgroup of $\mathbb{A}_{\mathbb{Q}}$.

(c) Prove that the quotient $\mathbb{A}_{\mathbb{Q}}/\Delta(\mathbb{Q})$ is compact.

Solution:

(a) Consider a set of continuous maps $f_i\colon X_i \to Y_i$ indexed by $i \in I$ and consider the product of these maps

$$f\colon \prod_{i \in I} X_i \to \prod_{i \in I} Y_i,\ f(x_i) = (f_i(x_i)).$$

We have $\pi_i \circ f(x_i) = f_i(x_i) = f_i(\pi_i(x))$, so the map f is continuous by the universal property of the product.

For each finite set of primes S, define

$$\mathbb{A}_{\mathbb{Q}}^S := \{(x_\infty, x_p) \in \mathbb{A}_{\mathbb{Q}} | \forall p \notin S : x_p \in \mathbb{Z}_p\} = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p.$$

The open subsets $\mathbb{A}_{\mathbb{Q}}^S \times \mathbb{A}_{\mathbb{Q}}^S$ cover $\mathbb{A}_{\mathbb{Q}} \times \mathbb{A}_{\mathbb{Q}}$ thus it suffices to prove that the multiplication map

$$\mathbb{A}_{\mathbb{Q}}^S \times \mathbb{A}_{\mathbb{Q}}^S \to \mathbb{A}_{\mathbb{Q}}^S$$

is continuous. In this case, it can be written as a product of the multiplication maps and it follows from the first step of the argument.

(b) Consider the open subset

$$U := \{(x_\infty, x_p) \in \mathbb{A}_{\mathbb{Q}} : |x_\infty| < 1, x_p \in \mathbb{Z}_p\}.$$

Let $x \in U \cap \Delta(\mathbb{Q})$ and suppose $x \neq 0$. Then $x = n/m$ for co-prime $n, m \in \mathbb{Z}$ and $m > 1$. There is a prime p with $p|m$. Then $|n/m|_p = |m|_p^{-1} > 1$. This is a contradiction. Thus $U \cap \Delta(\mathbb{Q}) = \{0\}$.

This implies that the set $\{0\} \subset \Delta(\mathbb{Q})$ is open. Therefore $\{x\} \subset \Delta(\mathbb{Q})$ is open because we can translate $\{0\}$ to $\{x\}$ with a homeomorphism of $\Delta(\mathbb{Q})$. Hence $\Delta(\mathbb{Q})$ is discrete.

We provide two proofs for the closeness of $\Delta(\mathbb{Q})$ inside the adeles. Let $x \in \mathbb{R}$ and $N \geqslant 1$. There exists an open neighborhood $x \in U \subset \mathbb{R}$ such that each $y \in U \cap \mathbb{Q}$ satisfies either $y = x$ or $y = m/n$ with m, n co-prime, $m \neq 0$, and $n \geqslant N$. Indeed, consider the set

$$Z := \{m/N : m \in \mathbb{Z}\} \subset \mathbb{Q}.$$

For any point $x \in \mathbb{R}$ there exists an open neighborhood $U \subset \mathbb{R}$ such that $U \cap Z \subset \{x\}$. The open subset U satisfies the assumptions of the claim.

Consider an adele $x = (x_\infty, x_p) \in \mathbb{A}_{\mathbb{Q}} - \Delta(\mathbb{Q})$. Let U be an open subset as above with $x = x_\infty$ and N large enough. Define the open subset

$$V := \{(y_\infty, y_p) | y_\infty \in U_{x_\infty},\ y_p \in x_p\mathbb{Z}_p\}.$$

Let $y \in V \cap \Delta(\mathbb{Q})$. There are $n, m \in \mathbb{Z}$ co-prime with $m > N$ such that $y = \Delta(n/m)$. Let $M := \max\{p : x_p \notin \mathbb{Z}_p\}$. Note that all the prime factors $p|m$ satisfy $p \leqslant M$ because $x_p \in \mathbb{Z}_p$ for all $p > M$. Write $m = \prod_{p \leqslant M} p^{e_p}$, then we have

$$\sum_{p \leqslant M} e_p \log(p) > \log(N).$$

Thus there is a prime $q \leqslant M$ such that

$$e_q \log(q) > \frac{\log(N)}{M}$$

and hence

$$e_q > \frac{\log(N)}{M \log(q)}.$$

Note that M is independent of N, so if we pick N large enough we can ensure

$$\frac{\log(N)}{M \log(p)} > \max\{v_p(x_p) : p \leqslant M\}.$$

for all primes $p \leqslant M$. This implies

$$e_q > v_q(x_q),$$

which is a contradiction to $y \in V$.

The second proof is based on the following general fact: Let $\Gamma \subset G$ be a subgroup of a Hausdorff group G such that Γ is discrete with the subspace topology. Then Γ is closed. Because Γ is discrete, there is an open neighborhood $U \subset G$ of the identity such that $U \cap \Gamma = \{e\}$. Because G is a topological group, we can find an open neighborhood of the identity $V \subset U$ that $VV^{-1} \subset U$. Let $x \in G - \Gamma$ and put $W := xV$. Consider $\gamma_1, \gamma_2 \in \Gamma \cap W$. Then there are $v_1, v_2 \in V$ such that $\gamma_1 = xv_1$ and $\gamma_2 = xv_2$. This implies

$$\gamma_2\gamma_1^{-1} = v_1v_2^{-1} \in U \cap \Gamma.$$

Thus $\gamma_1 = \gamma_2$, hence $|W \cap \Gamma| \leqslant 1$. If there exists $\gamma \in W \cap \Gamma$ then the Hausdorff property implies that there is an open neighborhood $W' \subset G$ such that $x \in W'$ and $\gamma \notin W'$. Thus $W' \cap W$ does not intersect Γ, contains x, and is open.

(c) Consider the subset

$$W := \{(x_\infty, x_p) \in \mathbb{A}_\mathbb{Q} : |x_\infty| \leqslant 1/2, x_p \in \mathbb{Z}_p\}.$$

Let $y \in \mathbb{A}_\mathbb{Q}$. I claim that there is $a \in \mathbb{Q}$ such that $y - a \in W$. Indeed, there is a finite set of primes S such that $y_p \in \mathbb{Z}_p$ for all $p \notin S$. If $p \in S$, there are $z_p, x_p \in \mathbb{Z}$ such that

$$y_p - \frac{z_p}{p^{x_p}} \in \mathbb{Z}_p.$$

Define $r := \sum_{p \in S} z_p/p^{x_p}$, then there is $b \in \mathbb{Z}$ such that $|y - r - b| \leqslant 1/2$. Thus

$$y - r - b \in W.$$

This implies that the map

$$W \to \mathbb{A}_\mathbb{Q}/\Delta(\mathbb{Q})$$

is a continuous surjection from a compact space. Thus the quotient is compact.

8. Show that for every $x \in \mathbb{Q}_p$ there exist unique $a_n \in \{0, \dots, p-1\}$ for $n \geqslant v_p(x)$ such that

$$x = \sum_{n \geqslant v_p(x)} a_n p^n.$$

Solution: Let $n \geqslant 1$ and $x \in \mathbb{Z}/p^n\mathbb{Z}$. There exist unique $a_0, \dots, a_{n-1} \in \{0, \dots, p-1\}$ such that $x = \sum_{i=0}^{p-1} a_i p^i$. Indeed, consider the map

$$\{0, \dots, p-1\}^n \to \mathbb{Z}/p^n\mathbb{Z},\ (a_0, \dots, a_{n-1}) \mapsto \sum_{i=0}^{n-1} a_i p^i.$$

To prove the claim is equivalent to proving that this map is bijective. The target and the source of this map have the same cardinality, so it suffices to prove that the map is injective. Suppose

$$\sum_{i=0}^{p-1} a_i p^i \equiv \sum_{i=0}^{n-1} b_i p^i \mod p^n$$

for $a_i, b_i \in \{0, \ldots, p-1\}$. Then

$$\sum_{i=0}^{n-1} (a_i - b_i) p^i \equiv 0 \mod p^n.$$

Suppose there is an index $0 \leqslant i < n$ with $a_i - b_i \neq 0$. Let $0 \leqslant j < p$ be the minimal index with this property. The above equation implies

$$p^j (a_j - b_j) \equiv 0 \mod p^{j+1}$$

and hence

$$a_j - b_j \equiv 0 \mod p.$$

This implies $a_j - b_j = 0$. This is a contradiction, hence the map is injective.

Consider $x = (x_n) \in \mathbb{Z}_p$. Write

$$x_n = \sum_{i=0}^{n-1} x_{n,i} p^i.$$

The uniqueness of the power series expansion implies

$$x_{n,i} = x_{n+1,i}$$

for all $n > i \geqslant 0$. Put $a_i := x_{i+1,i}$ for each $i \geqslant 0$, then we get

$$x_n = \sum_{i \geqslant 0} a_i p^i$$

and hence

$$x = \sum_{i \geqslant 0} a_i p^i.$$

The a_i are unique because they are unique modulo p^{i+1}.

Let $x \in \mathbb{Q}_p$. Then $p^{-v_p(x)} x \in \mathbb{Z}_p$ admits a unique power series expansion. This implies x admits a unique power series expansion.

9. Using Exercise 8, we define the fractional part of a p-adic number $x \in \mathbb{Q}_p$ to be

$$\{x\} := \sum_{0 > n > v_p(x)} a_n p^n \in \mathbb{Q}.$$

where $x = \sum_{n \geqslant v_p(x)} a_n p^n$. Define the map $\chi \colon \mathbb{A}_{\mathbb{Q}} \to \mathbb{T}$ by

$$\chi(x_\infty, x_p) := e^{2\pi i \{x_\infty\}} \prod_p e^{2\pi i \{x_p\}}.$$

(a) Prove that χ is a continuous character on $\mathbb{A}_{\mathbb{Q}}$.

(b) Define the character $\chi_a(x) := \chi(ax)$ for all $a, x \in \mathbb{A}_{\mathbb{Q}}$. Show that the map

$$\mathbb{A}_{\mathbb{Q}} \to \widehat{\mathbb{A}_{\mathbb{Q}}},\ a \mapsto \chi_a$$

is an isomorphism of topological groups.

Solution:

(a) Let S be a finite set of primes, then it is sufficient to prove that the restriction of χ to $\mathbb{A}_{\mathbb{Q}}^S$ is continuous. For each $x \in \mathbb{A}_{\mathbb{Q}}^S$, we get

$$\chi(x_\infty, x_p) = e^{2\pi i \{x_\infty\}} \prod_{p\in S} e^{2\pi i \{x_p\}}$$

Thus the restriction factors as

$$\mathbb{A}_{\mathbb{Q}}^S \to \mathbb{R} \times \prod_{p\in S} \mathbb{Q}_p \to \mathbb{T} \times \prod_{p\in S} \mathbb{T} \to \mathbb{T}$$

where the second map is the "product" of the local characters $x \mapsto e^{2\pi i\{x\}}$. The first map is continuous because the projections from the adeles to any of the factors is continuous. The second map is continuous because of the first step in Exercise 6. The last map is the addition map, so it is continuous because $\mathbb{T}$ is a topological group.

(b) Let $\chi \in \widehat{\mathbb{A}_{\mathbb{Q}}}$. Define

$$\chi_p(x) := \chi(0, 0, \ldots, 0, x, 0, \ldots)$$

for all $x \in \mathbb{Q}_p$ and

$$\chi_\infty(x) := \chi(x, 0, \ldots)$$

for all $x \in \mathbb{R}$. These are characters, so Exercise 5 implies that there is $\alpha_p \in \mathbb{Q}_p$ with $\chi_p(x) = e^{2\pi i \alpha_p x}$ and Exercise 2 implies that there is $\alpha_\infty \in \mathbb{R}$ with $\chi_\infty(x) = e^{2\pi i \alpha_\infty x}$. Consider the restriction

$$\chi_P \colon \prod_p \mathbb{Z}_p \to \mathbb{T},\ (x_p) \mapsto \chi(0, x_p).$$

This is a continuous character. Note that its image is closed and totally disconnected, so it is finite. This implies that the kernel is open and therefore there is a finite subset S of the primes such that

$$\{(x_p) \in \prod_p \mathbb{Z}_p | \forall p \in S : x_p = 1\} \subset \ker(\chi_P).$$

This implies $\alpha_p \in \mathbb{Z}_p$ for all $p \notin S$. Thus $(\alpha_\infty, \alpha_p) \in \mathbb{A}_{\mathbb{Q}}$. This implies that the map is surjective. Note that χ is uniquely determined by the χ_p and χ_∞. Thus Exercise 2 and 5 imply that the map is bijective.

Consider the subset $U := \{\chi \in \widehat{\mathbb{A}_\mathbb{Q}} : \chi_p|_{\mathbb{Z}_p} = 1\}$. The argument at the end of exercise 5 implies

$$\widehat{\mathbb{A}_\mathbb{Q}} = \bigcup_{\substack{(n_p)_p \in \prod_p \mathbb{N} \\ n_p \neq 0 \text{ for only finitely many primes } p}} \left(\prod_p p^{-n_p}\right) \cdot U.$$

The group on the left is locally compact and the union is countable. Thus Baire's category theorem implies that U contains an open subset. Since it is a subgroup, it is an open subset. Consider $n_p \in \mathbb{N}$ indexed by the primes such that only finitely many are non-zero. The set $\left(\prod_p p^{-n_p}\right) \cdot U$ is precisely the image of the open subset

$$\mathbb{R} \times \prod_p p^{-n_p}\mathbb{Z}_p \subset \mathbb{A}_\mathbb{Q}$$

under the map in the exercise. Since these sets form a neighborhood basis of the identity, this implies that Ψ is open and hence a homeomorphism.