<u>Example</u> The Gelfand spectrum of $\ell^\infty(\mathbb{N})$. We indicate the main steps of the identification of $\widehat{\ell^\infty(\mathbb{N})}$ as the set of all ultrafilters on $\mathbb{N}$. Details are left to the reader.

<u>Def 1.</u> A filter on $\mathbb{N}$ is a subset $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ with

(1) $\emptyset \notin \mathcal{U}$, $\mathbb{N} \in \mathcal{U}$

(2) $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$

(3) $A \subset B$, $A \in \mathcal{U} \Rightarrow B \in \mathcal{U}$.

It is called an ultrafilter if

(4) $A \in \mathcal{P}(\mathbb{N})$ then either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

<u>Examples 2.</u> (1) $\forall n \in \mathbb{N}$, $\mathcal{F}_n = \{A \subset \mathbb{N} : n \in A\}$ is an ultrafilter called principal ultrafilter.

(2) $\mathcal{F}_{\text{frechet}} = \{ A \subset \mathbb{N} : A^c \text{ is finite} \}$

is a filter called Frechet filter.

A convenient way to construct filters is via the finite intersection property. A subset $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ has the finite intersection property if whenever $A_1, \dots, A_n \in \mathcal{A}$, $\bigcap_{i=1}^{n} A_i \neq \phi$. One verifies then that

$$\mathcal{F} = \{ A \subset \mathbb{N} : \exists A_1, \dots, A_n \text{ in } \mathcal{A} \text{ with } A \supset A_1 \cap \dots \cap A_n \}$$

is a filter. In fact:

<u>Lemma 3</u> Let $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$. There is a filter $\mathcal{F} \supset \mathcal{A} \iff \mathcal{A}$ has the finite inters. property.

Of course filters are ordered by inclusion.

<u>Thm 4.</u> $\mathcal{F}$ is a maximal filter $\Longleftrightarrow$ $\mathcal{F}$ is an ultrafilter.

<u>Proof</u>: ($\Rightarrow$) Assume $\mathcal{F}$ maximal and $A \notin \mathcal{F}$. Since $\mathcal{F} \cup \{A\}$ is not a filter it does not have the finite intersection property. So there is $B \in \mathcal{F}$ with $B \cap A = \phi$, hence $B \subset A^c$ and hence $A^c \in \mathcal{F}$ since $\mathcal{F}$ is a filter. $\boxtimes$

Using Zorn's lemma one shows easily

<u>Lemma 5</u> Every filter is contained in a maximal one.

<u>Ex. 6</u>: In particular $\mathcal{F}_{\text{frechet}}$ is contained in an ultrafilter, and such ultrafilter is never principal.

A convenient way of describing ultrafilter is via a set function $\omega : \mathcal{P}(\mathbb{N}) \longrightarrow \{0,1\}$ such that (1) $\omega(\phi) = 0$, $\omega(\mathbb{N}) = 1$

(2) if A_1, A_2 are disjoint

$$\omega(A_1 \cup A_2) = \omega(A_1) + \omega(A_2)$$

Then $\mathcal{F}_\omega := \{ A \subset \mathbb{N} : \omega(A) = 1 \}$ is an ultrafilter and conversely every ultrafilter is of this form.

Let X be compact Hausdorff and ω an ultrafilter on $\mathbb{N}$.

<u>Lemma 7</u>: For every $f : \mathbb{N} \longrightarrow X$ there is a unique point denoted $x = \lim\limits_{\omega} f \in X$ such that: $\forall\ V \ni x$ open neighborhood,

$$\omega(f^{-1}(V)) = 1.$$

<u>Sketch of Proof</u>: Consider $\mathcal{F} := \{ F \subset X :$
F is closed and $\omega(f^{-1}(F)) = 1 \}$.

Show $\bigcap_{F \in \mathcal{F}} F$ is reduced to a point.

$\boxed{3}$

Let then $\mathcal{UF}(\mathbb{N})$ be the set of ultrafilters on $\mathbb{N}$. Then $\forall\, \omega \in \mathcal{UF}(\mathbb{N})$, $f \in \ell^\infty(\mathbb{N})$

$$\chi_\omega(f) := \lim_\omega f \in \mathbb{C}$$

is a character on $\ell^\infty(\mathbb{N})$.

<u>Thm. 8</u>: $\mathcal{UF}(\mathbb{N}) \longrightarrow \widehat{\ell^\infty(\mathbb{N})}$
$\omega \longmapsto \chi_\omega$

is a bijection.

<u>Hint</u>: let $\varphi \in \widehat{\ell^\infty(\mathbb{N})}$. Then:

$$\omega(E) := \varphi(\mathbb{1}_E), \quad E \subset \mathbb{N}$$

gives an ultrafilter on $\mathbb{N}$.