Now we turn to the next application of Thm 5.2: which is the spectral thm for a single normal operator $T \in B(\mathcal{H})$.

Let $T \in B(\mathcal{H})$ be normal, $A = \overline{\{ p(T, T^*) : p \in \mathbb{C}[X, Y] \}}$ the sub C^*-alg. gen. by T, and recall (see Prop. 4.10) that

$$Sp_A(T) = Sp(T)$$

where the latter is the sp of T as element of $B(\mathcal{H})$. Then the evaluation map $\hat{A} \to \mathbb{C}$, $\chi \mapsto \chi(T)$ induces a homeo $ev : \hat{A} \xrightarrow{\sim} Sp(T)$ (Thm 4.11) and for every $f \in C(Sp(T))$ there is a ! $f(T) \in A$ with

$$\chi(f(T)) = f(\chi(T)) \quad \forall \chi \in \hat{A}.$$

The map
$$C(Sp(T)) \to A$$
$$f \mapsto f(T)$$

is then a C^*-isom. sending $\mathbb{1}$ to $id_{\mathcal{H}}$ and $id: \lambda \mapsto \lambda$ to T. (Thm 4.11).

Then via $ev: \hat{A} \xrightarrow{\sim} Sp(T)$ we can transport the resol. of Id given by Thm 5.20 to one on $Sp(T)$ and we obtain:

<u>Corollary 5.24+25</u>: Let T = normal bdd and A as above.

(1) $\exists!$ resol. of id E on $Sp(T)$ s.t.

$$T = \int_{Sp(T)} \lambda \, dE$$

that is:

$$\langle Tx, y \rangle = \int_{Sp(T)} \lambda \, dE_{x,y}(\lambda) \quad \forall x, y \in \mathcal{H}.$$

(2) We have a commutative ~~The isomorphism~~ diagr. ~~extends to a~~

$$\begin{tikzcd} C(Sp(T)) \arrow[r] \arrow[d] & A \arrow[d] \\ L^\infty(E) \arrow[r, "\Psi"'] & B \end{tikzcd}$$

Where Υ is given by Thm 5.18.

Writing $f(T) = \Upsilon(f)$, $f \in L^\infty(E)$,

we have

$$f(T) = \int_{Sp(T)} f \, dE$$

that is

$$\langle f(T)x, y \rangle = \int_{Sp(T)} f(\lambda) \, dE_{x,y}(\lambda)$$

(iii) $B = \overline{\Upsilon(f(T))}$

(iv) If $\phi \subsetneq \omega \subset Sp(T)$ open then

$E(\omega) \neq 0$.

(v) $[S, T] = 0$ & $[S, T^*] = 0$

$\iff$ S commutes with $E(\omega)$

$\forall \omega \subset Sp(T)$, Borel.

Cor. 5.26: Let $T \in \mathcal{B}(\mathcal{H})$ be normal and $\lambda_0 \in Sp(T)$. Then

$$E(\{\lambda_0\}) = pr_{\mathcal{H}_{\lambda_0}}.$$

Proof:

(1) In the notations of Cor. 5.24+25:

$$TE(\lambda_0) = (id\,\chi_{\{\lambda_0\}})(T) \quad \text{and hence}$$

$$\langle TE(\lambda_0)x, y\rangle = \int_{Sp(T)} \lambda\, \chi_{\{\lambda_0\}}(\lambda)\, dE_{x,y}(\lambda)$$

$$= \lambda_0\, E_{x,y}(\{\lambda_0\})$$

$$= \lambda_0 \langle E(\{\lambda_0\})x, y\rangle$$

Thus $(T - \lambda_0 Id)E(\lambda_0) = 0$ and

$$Im\, E(\lambda_0) \subset \mathcal{H}_{\lambda_0}.$$

(2) Let $x \in \mathcal{H}_{\lambda_0}$: then $Tx = \lambda_0 x$ and $T^* x = \bar{\lambda}_0 x$. Thus $\forall p \in \mathbb{C}[X, Y]$

$$p(T, T^*)x = p(\lambda_0, \bar{\lambda}_0)x.$$

But $\lambda \mapsto p(\lambda, \bar{\lambda}_0)$ on $S_p(T)$

is norm dense in $C(S_p(T))$ hence

$$f(T)x = f(\lambda_0)x \quad \forall f \in C(S_p(T)).$$

This implies

$$\langle f(T)x, y\rangle = \int f(\lambda)\, dE_{x,y}(\lambda)$$

$$= f(\lambda_0)\langle x, y\rangle$$

$\Rightarrow$ $E_{x,y} = \langle x, y\rangle \delta_{\lambda_0}$ and

$$E_{x,y}(\{\lambda_0\}) = \langle x, y\rangle.$$

$$\langle E(\lambda_0)x, y\rangle \Rightarrow E(\lambda_0)x = x$$

$\forall x \in \mathcal{H}_{\lambda_0}$

$\Rightarrow \mathcal{H}_{\lambda_0} \subset \operatorname{Im}(E(\lambda_0)).$ $\blacksquare$