Overview

Abstract: In this seminar we will define quasimorphisms and use them as an algebraic tool to study various automorphism groups of manifolds. After a short introduction to symplectic geometry, we will mainly focus on the group of Hamiltonian diffeomorphisms and the Calabi quasimorphism.

Objective: By giving two half-hour talks, typing short summaries for those talks and participating in talks by others, each participant will get familiar with the concept of quasimorphism, learn about some concrete examples in and outside the world of symplectic geometry, as well as develop presentation and collaboration skills.

Organisation: Each student gives two talks, one in the first half, and one in the second half of the semester. On Fridays, the student giving the talk on the following Tuesday should discuss their first draft of the talk with Giovanni or Valentin. A short summary of the main definitions and statements should be typed into the common overleaf project by the day of the talk.

Prerequisites: Differential Geometry I. Background in Symplectic Geometry (as the course HS2023 in Symplectic Geometry) is very helpful but not strictly necessary.

Talk 0 - 20.2.:

Topic: **Automorphism Groups and Quasimorphisms.**
Speakers: Giovanni Ambrosioni, Valentin Bosshard
Content:

- Definition of various automorphism groups.
- Some known theorems about automorphism groups with focus on simplicity and perfectness.
- Quasimorphisms: what they are and why they are useful to study automorphism groups.

Part One

Talk 1 - 5.3. (*easy*):

Topic: **Introduction to Quasimorphisms.**
References: [Car13, Chapter 3.1], [PR14, Chapter 3.1], [GG04, Introduction]
Content:

- Definition of quasimorphisms, defect, homogeneous quasimorphisms.
- Explain the bijection between the space of quasimorphisms modulo bounded functions and homogeneous quasimorphisms (see e.g. [Car13, Proposition 3.1.4]).
- Definition of simple and perfect groups, commutator and commutator length.
- Show that if a non-trivial homogeneous quasimorphism exists then the commutator length is unbounded (see e.g. [GG04, Introduction]).
- Give a few examples of automorphism groups which are simple/not simple, perfect/not perfect.

Mentor: Giovanni

TALK 2 - 12.3. (*medium*):

Topic: **The Homeomorphism group of the circle.**
References: [Man15, Chapter 1.3], [Ghy01, Chapter 4]
Content:

- State the theorem that $\mathrm{Homeo}_c(M)_0$ is simple and perfect (we completely prove it for $M = S^1$ in this talk).
- Indicate the general strategy for proving simplicity and perfectness of automorphism groups.
- Explain the fragmentation property.
- A connected topological group is generated by a neighborhood of the identity.
- The fragmentation property of a homeomorphism group implies perfect and simple [Man15, Corollary 1.9,1.10].
- Conclude that $\mathrm{cl}(\mathrm{Homeo}_c(\mathbb{R}^n)) = 1$.
- Definition of $\mathrm{Homeo}_0(S^1)$. Why is $\mathrm{Homeo}_0(S^1) = \mathrm{Homeo}_+(S^1)$, where the latter denotes the group of orientation-preserving homeomorphisms? (Look at graph of the maps)
- $\mathrm{Homeo}_0(S^1)$ has the framentation property for three intervals [Ghy01, the last paragraph of the proof of Theorem 4.3].
- Conclude that we proved $\mathrm{Homeo}_0(S^1)$ is perfect and simple.

Mentor: Valentin

TALK 3 - 19.3. (*easy*):

Topic: **Rotation and translation numbers.**
References: [Ghy01, Chapter 5]
Content:

- Definition of the universal cover of an automorphism group and proof that the definition given for $\widetilde{\mathrm{Homeo}}_0(S^1)$ in [Ghy01, beginning of Chapter 4] is equivalent.
- What topology do we consider on these groups?
- Definition of the translation number and proof that it is a quasimorphism.
- Prove that $\mathrm{frag}(\mathrm{Homeo}_0(S^1)) = 2$ and conclude that $\mathrm{cl}(\mathrm{Homeo}_c(S^1)) \leq 2$.
- The translation number is the only quasimorphism (Proposition 5.4).
- Definition of the rotation number.
- If time allows: The rotation number is rational iff there is a periodic point [Mer19, Proposition 16.1].

Mentor: Valentin

TALK 4 - 26.3. (*medium*):

Topic: **Quasimorphisms on the automorphism group of the disc.**
References: [GG04, sections 2.1, 5.1 and 5.2]
Content:

- Definition of $\mathrm{Diff}_0(D^2, \partial D^2, \mathrm{area})$.
- Explain the construction of Ruelle's invariant on $\mathrm{Diff}_0(D^2, \partial D^2, \mathrm{area})$ (section 2.1).
- Show that it is a non-trivial quasimorphism.
- Define the pure braid group $P_n(D^2)$ (section 5.1).
- Explain how a quasimorphism on $P_n(D^2)$ gives rise to a quasimorphism on $\mathrm{Diff}_0(D^2, \partial D^2, \mathrm{area})$ (section 5.2, see also [BKS18, pages 2 and 3]).

Mentor: Giovanni

TALK 5 - 9.4. (*medium*):

Topic: **Quasimorphisms on the automorphism groups of closed oriented surfaces.**
References: [GG04]
Content:

- Mention $\text{Diff}_0(\Sigma, \text{area})$ is not simple in general, there is a non-trivial homomorphism to $H_1(\Sigma)$, called Calabi homomorphism. Its kernel is simple, so we are actually interested in nontrivial quasimorphisms on this kernel.
- State the existence result [GG04, Theorem 1.2].
- Explain the construction of Ruelle's invariant on $\text{Diff}_0(T^2, \text{area})$ (section 2.2).
- Summarize the idea for constructing a quasimorphism on $\text{Diff}_0(\Sigma_g, \text{area})$ for $g \geq 2$ (section 2.3).

Mentor: Giovanni

Talk 6 - 16.4. (*easy*):

***Topic:* Introduction to symplectic geometry and the group of Hamiltonian diffeomorphisms.**
References: [MS17, Chapter 3], ([Pol12, Chapter 1], [Car13, Chapter 2.1]) .
Content:

- Definition of symplectic manifolds and Lagrangian submanifolds. Examples $\mathbb{R}^{2n}$ and surfaces.
- Statement of Darboux theorem without proof.
- Symplectic manifolds are orientable, even-dimensional.
- Definition of symplectomorphisms and $\text{Symp}(M)$.
- Symplectic vs Hamiltonian vector fields (repeat Cartan's formula for the Lie derivative).
- Definition of Hamiltonian diffeomorphisms and $\text{Ham}(M)$.
- Example of height function on sphere ([MS17, Example 3.1.7]).
- Proof that $\text{Ham}(M)$ is a normal subgroup of $\text{Symp}_0(M)$ ([MS17, Exercise 3.1.14]). Maybe subsection *Hamiltonian symplectomorphisms* until Exercise 10.1.3 also helps.
- Hofer metric on $\text{Ham}(M)$ and its relative version. Mention the diameter theorems and conjectures [MS17, Chapter 12.3 and p.480], in particular [KHA09].

Mentor: Valentin

Part Two

The contents of Talks 10 and 11 may be adjusted along the way.

Talk 7 - 23.4. (*easy*):

***Topic:* The Flux homomorphism.**
References: [MS17, Chapter 10.2], [Car13, Chapter 2.2],[Ban97, Chapter 3].
Content:

- Definition of exact symplectic manifolds.
- Characterization of Hamiltonian isotopies [MS17, Proposition 9.3.1, Corollary 9.3.3] or [Car13, Lemma 2.2.1]. Conclude with [MS17, Equation 10.1.7].
- Definition of the flux homomorphism $\widetilde{\text{Symp}}_0(M, \omega) \to H^1(M)$.
- Geometric interpretation of flux [MS17, Equation 10.2.2].
- Flux characterizes Hamiltonian isotopies ([MS17, Theorem 10.2.5]).
- Exact sequences induced by flux ([MS17, Proposition 10.2.13]).
- (More general definitions of the flux [Ban97, Chapter 3]).

Mentor: Valentin

Talk 8 - 30.4. (*easy*):

***Topic:* The Calabi homomorphism and Calabi quasimorphisms.**
References: [MS17, Chapter 10.3], [Car13, Chapter 2.2, 3.2].
Content:

- Definition Calabi homomorphism on $\text{Ham}_c(M)$ for exact symplectic manifolds M.
- Well-definedness, homomorphism property and uniqueness.

- $\text{Ham}_c(M)$ is not simple for (M, ω) exact, but the kernel of the Calabi homomorphism is simple ([Ban97] without proof).
- $\text{Ham}(M)$ is simple ([Ban97] without proof). This motivates the definition of Calabi quasimorphisms.
- Explain why the quasimorphisms from talks 4 and 5 are non-trivial on the kernel of the Calabi homomorphism resp. the flux homomorphism ([GG04], careful: the flux homomorphism is also called Calabi homomorphism in [GG04]).
- List possible requirements on the definition Calabi quasimorphisms that one might wish to have ([Py06b, page 178 starting at question in italic up to page 179] which refers to [EP03]).

Mentor: Valentin

TALK 9 - 7.5. (*medium*):

Topic: **Calabi quasimorphisms on $\text{Ham}(T^2)$.**
References: [Py06a] (in French!)
Content:

- Construct the homogeneous quasi-morphism C_ϕ (see second half of section 2 in [Py06a] and Talk 4)
- State Theorem 2.1 from [Py06a] and explain how this provides a construction for Calabi quasimorphisms.
- Prove it (see [Py06a, section 2]).
- Define $\mathcal{F}$ and the Reeb graph.
- State Theorem 0.2
- Outline its proof ([Py06a, section 3])).

Mentor: Valentin

TALK 10 - 14.5. (*hard*):

Topic: **Filtered Hamiltonian Floer homology & the Hofer metric**
References: [EP03], [AD14].
Content: Try to extract the general ideas/concepts for the key words listed below.
Informal introduction to FH: `http://jde27.uk/blog/fukaya1.html` (skip 'Fukaya'-stuff).

- Morse functions (Chapter 1 in [AD14]) and Morse homology (Chapter 3). (Instead of working with pseudogradient fields, we can work with a Riemannian metric). Mention why it is useful and Morse inequalities.
- Floer homology (section 2.5 in [EP03], Chapter 6 in [AD14]), Quantum homology (intro 2.3 and 2.3.1 in [EP03], more to come), their relation (sections 2.6.1-2.6.2 in [EP03], [PSS96, Theorem 4.1]). Mention a bit of history (Arnold conjecture, ... e.g. Section 6.1 in [AD14]).
- Spectral invariants (section 2.6.2 in [EP03]) (postpone to next talk?).

Mentor: Giovanni

TALK 11 - 21.5. (*hard*):

Topic: **A Calabi quasimorphism on $\text{Ham}(S^2)$ and applications.**
References: [EP03].
Content: Try to extract the general ideas/concepts for the key words listed below.

- Spectral invariants (section 2.6.2), construction of Calabi quasimorphism on $\text{Ham}(S^2)$ (Theorem 3.1, sections 3.4 and 3.5).
- Mention the formula for autonomous Hamiltonian diffeomorphisms (Theorem 5.4) and its relation to Hofer's metric.
- Explain some applications:
  (1) First possibility: Construction of 'controlled' (non-Calabi afair) homogeneous quasimorphism on symplectic balls, either in dimension 2 or in general (two dimentional version in [KHA09], arbitrary dimension in [Sey14]). Mention how this solves LHC for balls but not for S^2 (still open!).

(2) Second possibility: proof of non-displeaceability of the Clifford torus in projective spaces a' la [BEP04].

Mentor: Giovanni

Talk 12 - 28.5. (*easy*):

***Topic:* Construction of the real numbers by quasimorphisms.**
References: [AC21]
Content:

- Definition of real numbers via "slopes"
- Explain what it has to do with quasimorphisms, define addition, multiplication, integers.
- What are rational numbers, $\sqrt{2}$, root of a polynomial, π in this model?

Mentor: Giovanni

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