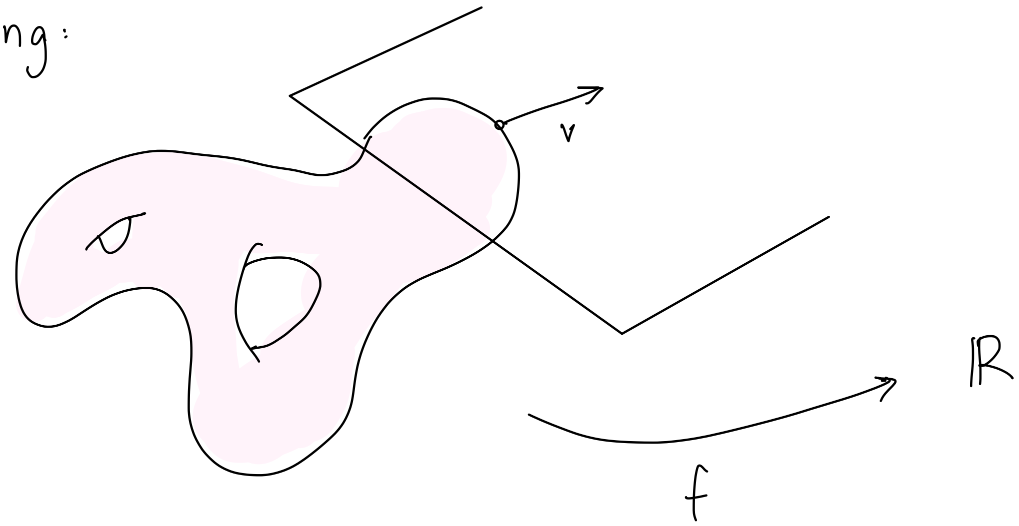
Differential Geometry II, FS2024

Last lecture: overview of the course

Beginning:



- M smooth manifold, $f: M \to \mathbb{R}$ smooth function
- T_pM tangent plane
- tangent vectors as derivations:

$$\vec{v}(f)(p) = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)), \qquad \gamma(0)=p \quad \gamma'(0)=\vec{v}$$

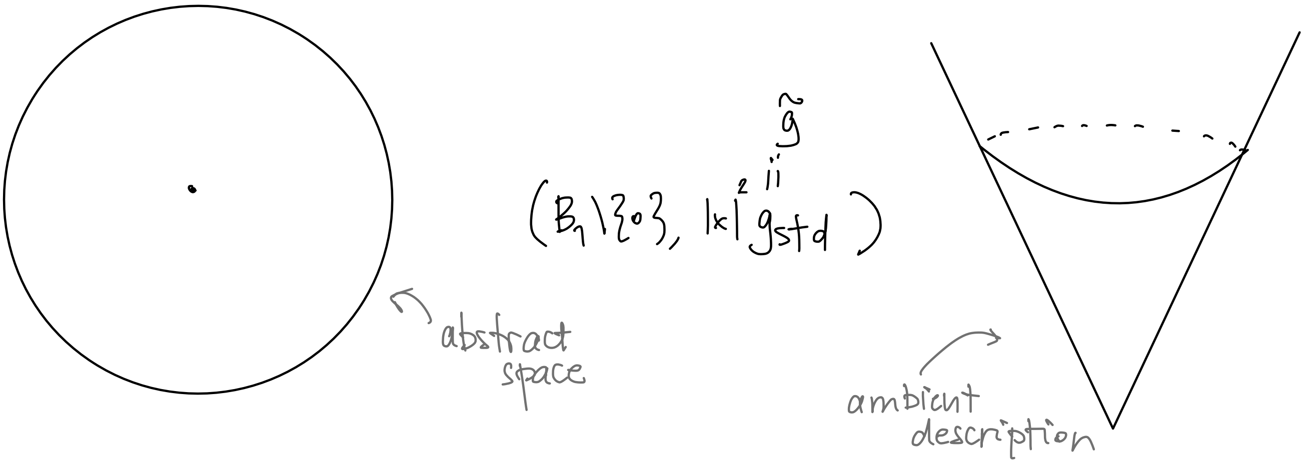
During the course:

1) additional structure was added to the picture above (metrics, connections, curvature tensors, ...)
2) how does the "abstract" abovementioned structure interact with the underlying picture? (volume, completeness, compactness, shape, ...)

1. Metrics A Riemannian manifold is a smooth manifold endowed with a metric g.

$$g \in \Gamma(T_{0,2}M) \text{ symmetric, positive definite}$$
(g_x is a scalar product on T_xM varying smoothly in x)

Rmk. in general we imagine objects with the ambient metric. For example if we put a different metric on the disk



Rmk. By Nash's embedding Theorem (1.20) every Riemannian manifold can be thought of as sitting in some $\mathbb{R}^L$.

$$(M,g) \text{ given} \Rightarrow \exists\, \underset{\text{(embedding)}}{F: M \to \mathbb{R}^L} \text{ s.t. } \underset{\text{(isometry)}}{F^* g_{\mathbb{R}^L} = g}$$

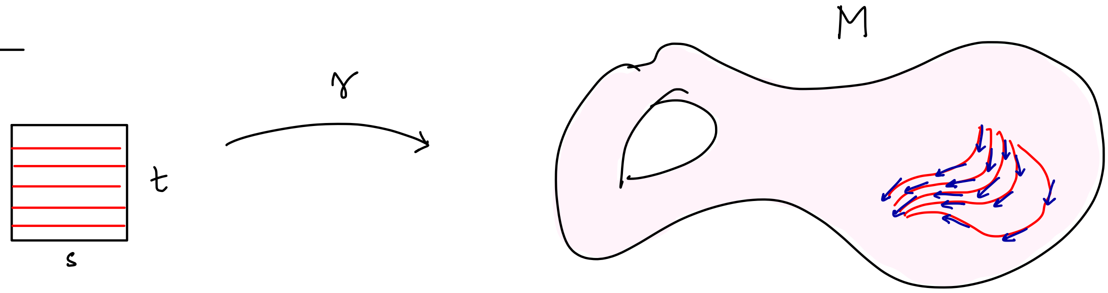
From the metric we obtain the notion of distance:

$$d(p,q) = \inf\left\{ L(\gamma) = \int_a^b |\gamma'(t)|_{g(\gamma(t))}\, dt,\ \begin{array}{l} \gamma: [a,b] \to M \\ \gamma(a) = p \quad \gamma(b) = q \\ \gamma \text{ piecewise } C^\infty \end{array} \right\}$$

(M,d) is a metric space with the same induced topology (Th. 2.3)

2. Connections

Motivation:



$$\partial_s \big(\underbrace{\partial_t \gamma(s,t)}_{\in T_{\gamma(s,t)}M} \big) \quad ?$$

How can we take derivatives of objects taking values into the "moving" planes $T_{\gamma(s,t)}M$?

Notice that these kind of derivatives arise "naturally" for example when trying to find the shortest red curve in the picture above.

Idea: vector fields are derivations of functions

connections are derivations of sections.

Definition (3.1) Given a vector bundle $\pi: E \to M$ on M, a connection is a bilinear map

$$\nabla : \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$$

covariant derivation of section is a section

satisfying $\nabla_V(fs) = (Vf)s + f\nabla_V s$, $\nabla_{fV} s = f\nabla_V s$.

When $E = TM$, there exists the Levi-Civita connection, uniquely determined by the metric and explicitly given by

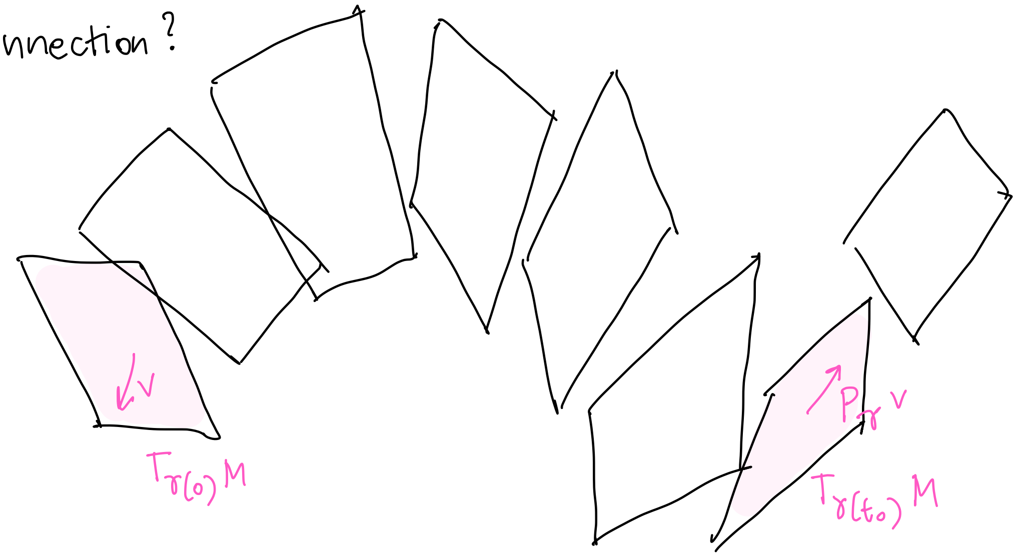
$$2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle + \langle Z,[X,Y]\rangle$$

Koszul formula, (Thm 3.7)

Rmk. most useful for computations is the use of Christoffel symbols

$$\Gamma_{ij}^{k} e_k = \nabla_{e_i} e_j$$

Why connection?



Consider the following plane distribution $T_{\gamma(t)}M$ along some curve γ.

A priori we only know $T_{\gamma(0)}M \cong \mathbb{R}^m \cong T_{\gamma(t_0)}M$, but thanks to the connection there are some "distinguished" choices for the isomorphisms.

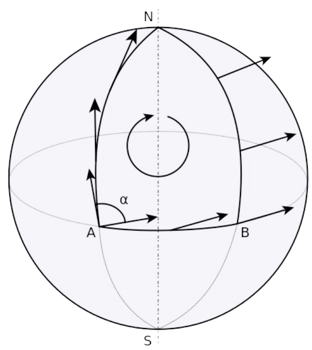
step 1: solve $\begin{cases} \nabla_{\dot\gamma} e^{\vec v} = 0 \\ e^{\vec v}(0) = \vec v \end{cases}$

step 2: $T_{\gamma(0)}M \xrightarrow{\sim} T_{\gamma(t_0)}M$

$\vec v \longrightarrow e^{\vec v}(t_0)$

is an isometry called the parallel transport.

parallel transport on the sphere



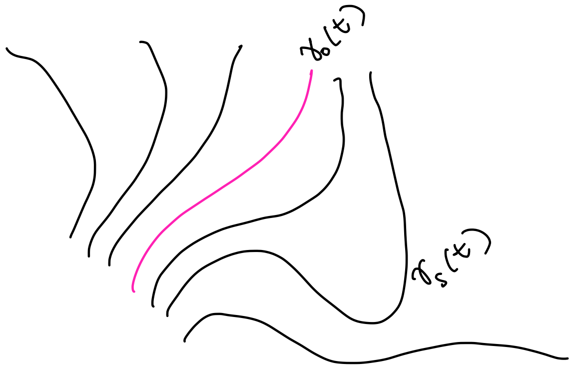
3. Geodesics

Definition (3.19)

$$\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0$$

$$\Leftrightarrow \quad \ddot\gamma^k + \Gamma_{ij}^k \dot\gamma^i \dot\gamma^j = 0$$

The motivation is the first variation of length formula



Given a family of geodesics $\gamma_s(t)$, $|\dot\gamma_s(t)| \equiv 1$,

Lemma 3.18

$$\frac{d}{dt} L(\gamma_s) = \langle V, \gamma' \rangle \Big|_0^1 - \int_0^1 \langle V(t), \gamma''(t) \rangle \, dt$$

$V(t) = \partial_s \gamma_s(t)$

change related to variation near end points

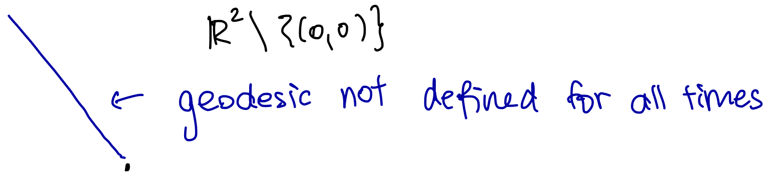
change related to the variation "away" from end points

Rmk. Given <u>any</u> vector field V along γ, we can build a $\gamma_s(t)$ such that $V(t) = \partial_s \gamma_s(t)$ (Exercise 4.1)

"geodesics have vanishing first variation of length in a loose at the ends sense"

Rmk. Given a point and an initial velocity, p, $\vec{v} \in T_pM$, there is always a geodesic with these initial conditions (ODE theory)

⚠ in general the geodesic will <u>not</u> be defined for all times



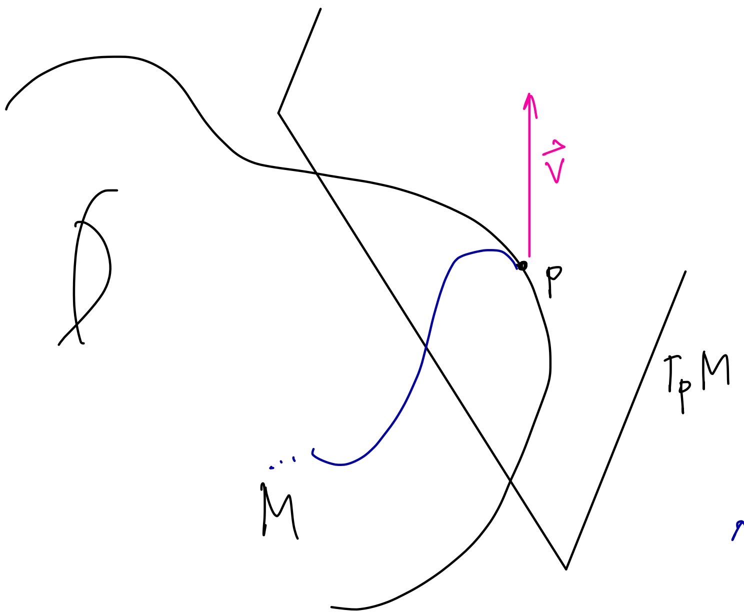
3.1. Local picture, exponential map

One can define a map that to a point p, a vector $\vec{v} \in T_pM$ and a time t assigns the point reached by the (unique) geodesic γ starting at p with velocity $\vec{v}$ after time t (in unit speed parametrization). Of course one can only do this operation for times t that are in the domain of existence of the solution to the geodesic equation with the given initial conditions.

This map is called the <u>exponential map</u> and it turns out that locally it can be used to describe well the manifold M:

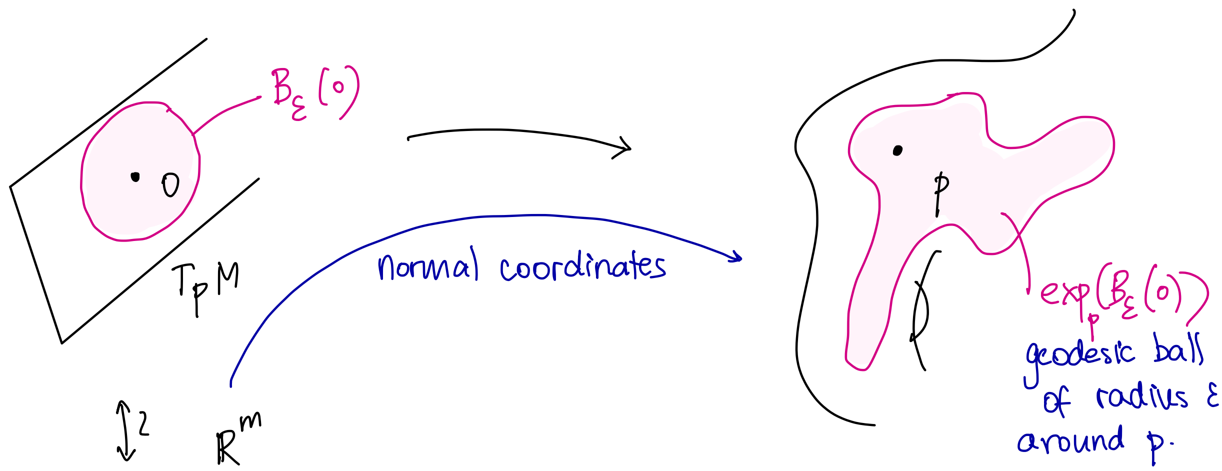
$$\begin{aligned} \exp : \quad D_1 \subset TM \times \mathbb{R} &\to M \\ ((p, \underset{|v|=1}{v}), t) &\mapsto \gamma_p^{\vec{v}}(t) \end{aligned}$$

(Def. 4.2)



$\gamma_p^{\vec{v}}(t) = \exp_p(t\vec{v})$

By Prop 4.4. $\exp_p$ is a diffeomorphism around 0:



In normal coordinates one can "read" the geometry of the mfd:

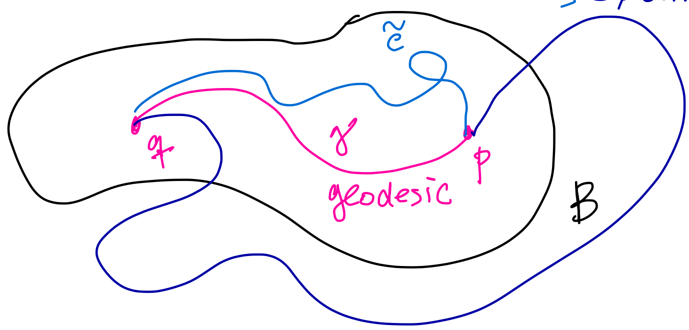
1) $g_{ij}(0) = \delta_{ij}(0)$ $\quad \partial_k g_{ij}(0) = 0 \quad \Gamma^k_{ij}(0) = 0$ (centering)

2) see later after curvature discussion

The geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$ has a strong connection to the first variation of the length (see above). As one expects, geodesics thus have many properties related to length-minimization.

1) <u>In every geodesic ball B:</u> Any geodesic contained in B is minimizing (Theorem 4.10)

$\tilde c$, c, competitor (can also escape B)



$L(\gamma) \le L(\tilde c), L(c)$

2) If a curve has the minimization property, then it is a geodesic

$$\left(\gamma(0)=p \quad \gamma(1)=q, \quad L(\gamma) \le L(\tilde\gamma) \quad \forall \tilde\gamma \text{ joining } p \text{ and } q \quad \Rightarrow \quad \nabla_{\dot\gamma}\dot\gamma = 0 \right)$$

3) (later) If the geodesic has a conjugate point, then it is not minimizing, otherwise it minimizes length among nearby curves.

Theorem 4.12 (Hopf-Rinow)

(M,d) complete $\Leftrightarrow$ exponential defined on entire TM

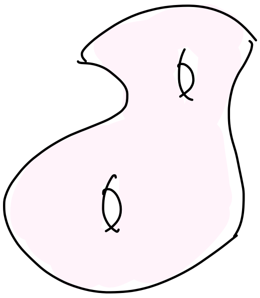
$\Leftrightarrow$ exponential defined on entire T_pM for one point p

$\Downarrow$ $\not\Uparrow$ for example $B_1^m(0)$

any two points can be connected by a minimizing geodesic

4. Curvature

In the first semester we saw an intrinsic notion of curvature.



$\Sigma \subseteq \mathbb{R}^3$

Gauss-Bonnet:

$$\int_\Sigma K = 2\pi \chi(\Sigma)$$

this is one example of curvature $\leftrightarrow$ topology interaction

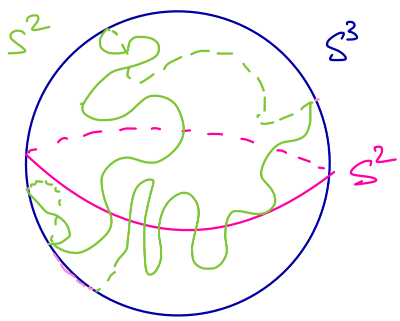
In differential geometry II we introduced many notions of curvature both intrinsic and extrinsic.

does not rely on the ambient space

does rely on the ambient space

"how does my object interact with the world it lives in?"

As an example consider two possible embeddings of S^2 in S^3. For the extrinsic notions of curvature there is a way to see that the pink S^2 is "special". From the intrinsic point of view one can apriori not tell.



4.1 Overview of the curvature tensors:

1) Riemann curvature tensor

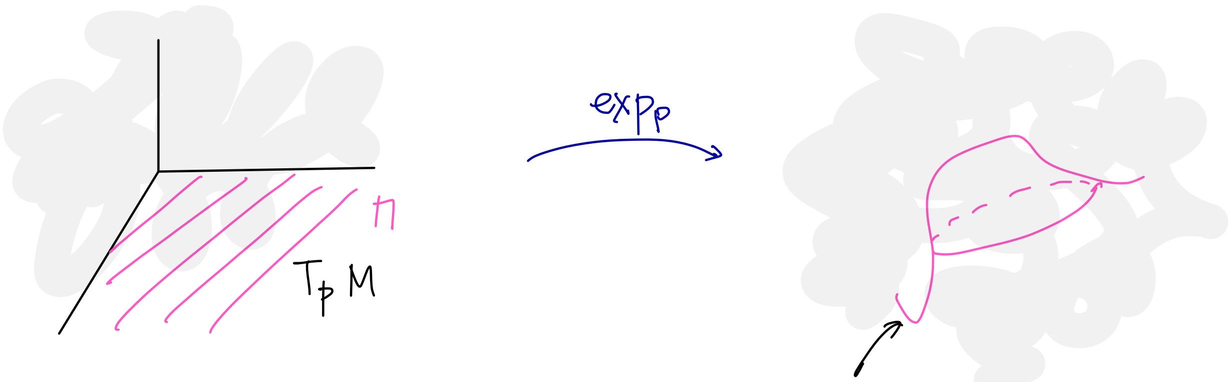
$$R(X,Y,Z,W) = \langle \underbrace{R(X,Y)}_{\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z} Z, W \rangle$$

- very big / complicated tensor   hard to use or make assumptions on
- interesting algebraic properties   two dimensional planes are the building blocks of curvature

2) Sectional curvature (Def. 5.5)

$$K(\Pi) = -\frac{R(X,Y,X,Y)}{|X \wedge Y|^2}$$

↪ turns out to be the Gauss curvature of the surface one obtains by applying the exponential map to the plane Π



2-d. surface in higher dim M^m
↪ has some intrinsic Gauss curvature
↪ this is the sectional curvature of the plane Π

Fact: (algebraic, see Lemma 5.8) If you know $K(\Pi)$ for all possible two dimensional planes in T_pM, then you actually know the entire Riemann curvature tensor.

In the case of constant sectional curvature κ_0 one can write down explicitly the curvature tensor: (Proposition 5.9)

$$R = \kappa_0 R' \qquad R'(X,Y,Z,W) = \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle$$

In general, the assumption of constant sectional curvature is very strong and we saw in chapter 7

Theorem 7.2 (Killing-Hopf)
(M,g) an m-dim. space form (complete, const. sectional curvature) then

$$(M,g) \underset{\text{isometric}}{\simeq} \mathbb{M}^m_k / \Gamma$$

isometry subgroup

$\mathbb{R}^m$, $\mathbb{S}^m_k$, $\mathbb{H}^m_k$

An interesting class of manifolds is the one of Cartan-Hadamard mfds (Ch 8) which is defined in terms of negative sectional curvature $K \leq 0$.

Theorem 8.1 (Cartan-Hadamard)
(M,g) (simply) connected and complete, $K(\Pi_p) \leq 0$ $\forall \Pi \subseteq T_pM$ $\forall p \in M$, then $\exp_p : T_pM \to M$ is a covering map (diffeomorphism).

Interested? Have a look at the Cartan-Hadamard conjecture!
Roughly speaking one can think of negative sectional curvature as "same perimeter holds more volume than flat space"

A notion of curvature than contains less information than the entire Riemann curvature tensor is the "averaged sectional curvature" or Ricci curvature:

Definition 5.12 $\mathrm{Ric}(Y,Z) = \sum_{i=1}^{m} R(e_i, Y, Z, e_i)$

ONB

(0,2) tensor

$$\left(\mathrm{Ric}(Y,Y) = (m-1) \int_{\substack{\pi \text{ planes} \\ \text{cont. } Y}} K(\pi)\, d\pi\right.$$

Definition 5.18 scalar curvature:

$$\mathrm{scal}_g = \mathrm{tr}_g \mathrm{Ric}$$

real valued

$$= \sum_i \mathrm{Ric}_p(e_i, e_i) = m(m-1) \fint_{\substack{\pi \text{ plane} \\ \text{at } p}} K(\pi)\, d\pi$$

Since these quantities are averages of the previously discussed scalar curvature, we expect them to give us less information about the underlying spaces. In dimension 3, thanks to special algebraic properties it turns out that the Ricci tensor still contains the information about the entire Riemann curvature tensor. From dimension ≥ 4 no longer true

Exercise 5.1

(there exist ricci-flat metrics which are not flat...

you can google Schwarzschild metric or Calabi conjecture to find out...)

Some nice results related to Ric proven in class:

Theorem 5.29 i) $\mathrm{Ric}_g = fg$, $\dim(M) \geq 3 \Rightarrow f \equiv cst$

dim 2
↳ back to Gauss, less rigid

ii) $K_p(\pi) = cst$ for some point $p \in M$

$\Rightarrow \sec \equiv$ constant.

Theorem (Myers 1941) 6.15 (M,g) complete, connected, $\exists \kappa > 0$ s.t.

$$\mathrm{Ric}(X,X) \geq (m-1)\kappa |X|^2$$

$$\Rightarrow \quad \mathrm{diam}(M) := \sup_{p,q \in M} d(p,q) \leq \frac{\pi}{\sqrt{\kappa}}$$

In particular the theorem shows that a lower bound on the Ricci curvature forces compactness. Recall that negative sectional curvature (+ simply conn) implies that the exp map is a diffeomorphism!

4.2 Some formulas for intuition

in normal coordinates:

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{iklj}(p) x^k x^l + O(|x|^3)$$ "Riemann - metric"

$$\det(g_{ij}) = 1 - \frac{1}{3} \mathrm{Ric}_{kl}(p) x^k x^l + O(|x|^3)$$

"Ricci - volume"

$$\mathrm{Vol}(B_r^d(p)) = \omega_m r^m \left(1 - \frac{\mathrm{Scal}(p)}{6(m+2)} r^2 + O(|r|^3)\right)$$

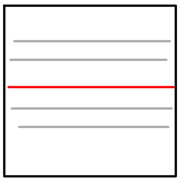
$$\mathrm{Length}(C_r) = 2\pi r - \frac{\pi}{3} K(\Pi_p) r^3 + O(r^4)$$

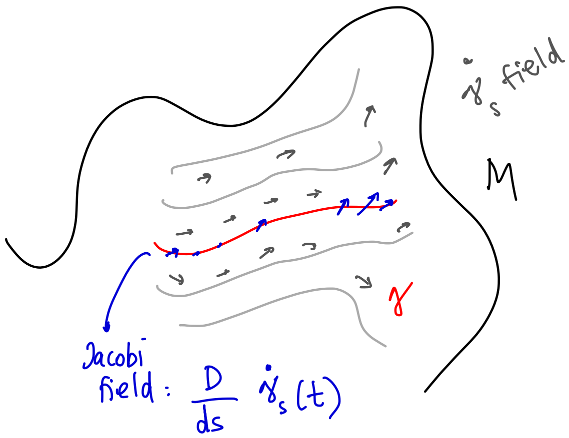
image of $S^1 \subseteq \Pi_p$
under exponential map

If there is time, extrinsic curvature

5. Jacobi fields, conjugate points, geodesics

Jacobi fields are special kinds of vector fields along geodesics γ. They are the ones that infinitesimally describe families of geodesics close to γ.





Equivalently, given a geodesic γ a Jacobi field $V(t) \in T_{\gamma(t)}M$ is a solution to the Jacobi equation (Lemma 6.1)

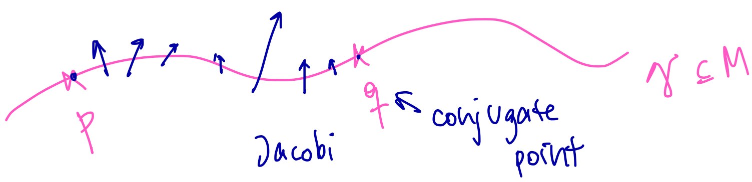
$$V'' + R(V, \gamma')\gamma' = 0 .$$

← Rmk: $V^{\perp}$ is also a Jacobi field

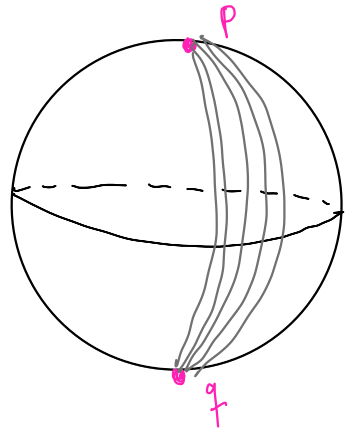
In the case of constant sectional curvature one can explicitly write solutions in terms of parallel frames along γ (see Example 6.4) as well as to understand curvature in terms of geodesic behaviour.

Here we emphasize the relation to conjugate points and geodesics:

<u>Definition 6.5.</u> Let γ be a geodesic, $\gamma(0)=p$, $\gamma(t_0)=q$. Then q is conjugate to p along γ if there exist a Jacobi field $J \not\equiv 0$, $J(0)=0$, $J(t_0)=0$ along γ.

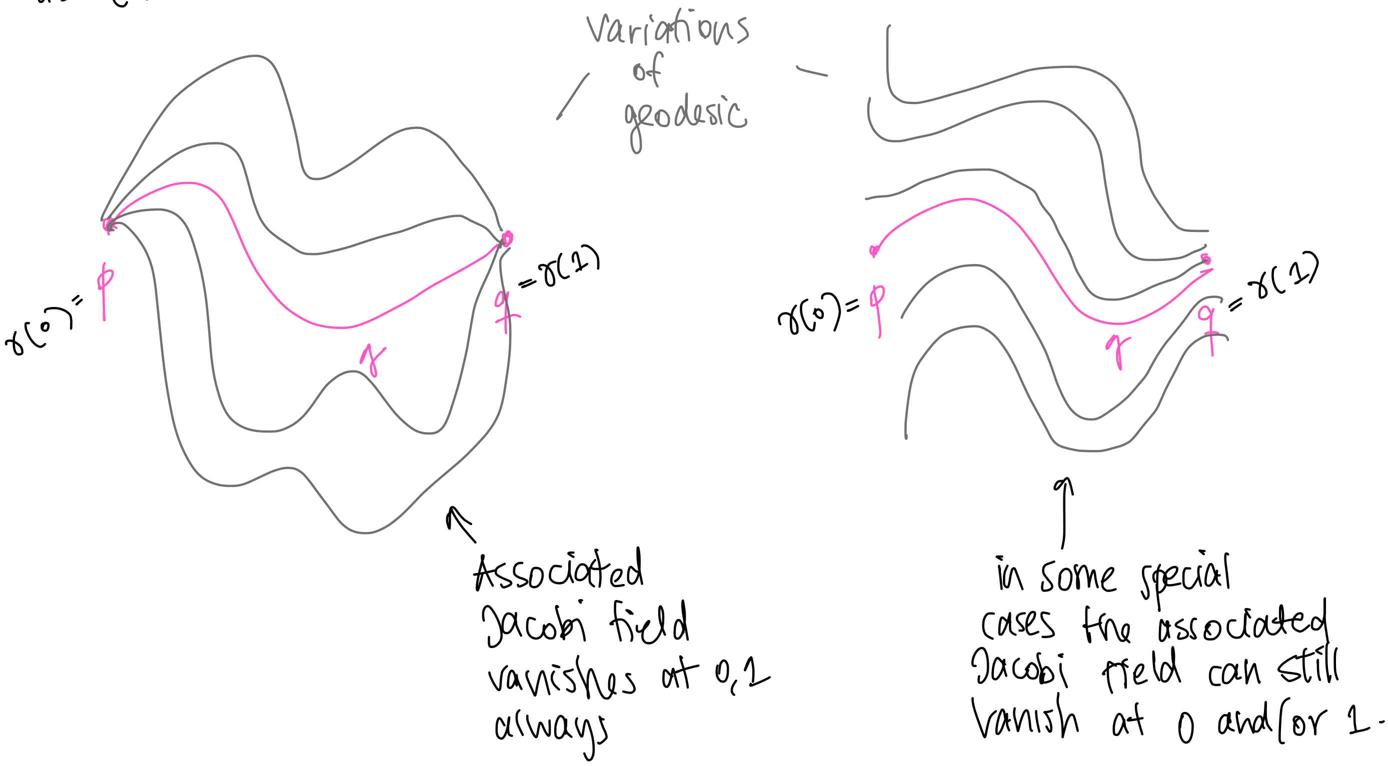


Example:



take the variation of geodesics coming from "turning"

Remark: having a Jacobi field with $J(p) = J(q) = 0$, does not necessarily implies that it comes from a variation of geodesics that fixed the points $\gamma_s(p)$ and $\gamma_s(q)$ like in the example above!



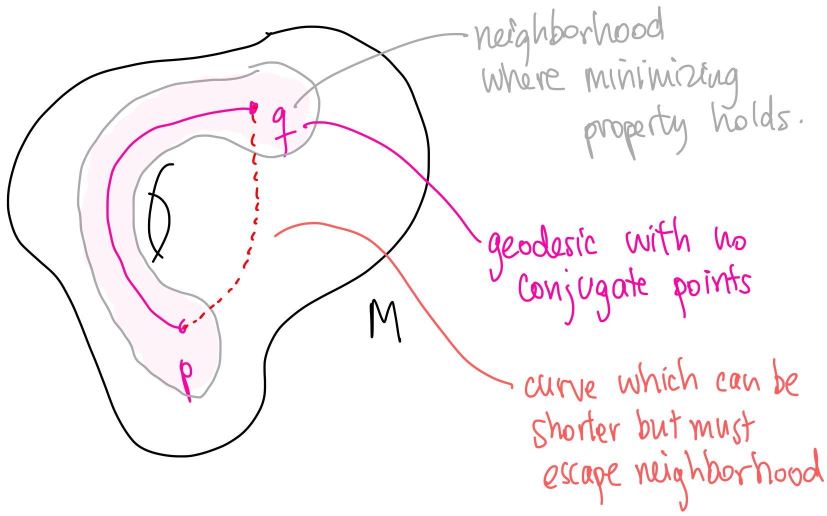
Lemma 6.7: $\gamma(t_0)$ conjugate to $\gamma(t) \Leftrightarrow d_{t_0 v}\exp_p : T_{t_0 v}(T_pM) \to T_{\gamma(t_0)}M$ not injective.

Exercise 8.1: if γ has a conjugate point and c is another geodesic which remains close to γ at any time, then c must also have a conjugate point.

Finally, let us recall the relationship between conjugate points and length minimizing properties of geodesics:

Theorem 6.8 Assume γ is a geodesic with no conjugate points. Then, among curves with the same endpoints that are in a neighborhood of γ, γ has minimal length:

$$L(\gamma) \leq L(c)$$



Theorem 6.12 Let $t_0 \in (0, \ell)$ be a conjugate point for $\gamma : [0, \ell] \to M$. Then γ is not the shortest curve from $\gamma(0)$ to $\gamma(\ell)$.

⚠ careful – the conjugate point is a bit before the point where minimality fails in general. Indeed there are examples where t_0 is a conjugate point for γ and there is a unique geodesic from $\gamma(0)$ to $\gamma(t_0)$ (which is γ).