1. Riemannian metrics, isometries, Lie derivatives

1.1. Equivalence of definitions of a surface.

Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface. Show that for all $p \in \Sigma$ there exists an open neighborhood $U \subset \mathbb{R}^3$ of p so that $\Sigma \cap U$ is the graph of a smooth function $F\colon V \subset \mathbb{R}^2 \to \mathbb{R}$, i.e. $\Sigma \cap U = \{(x^1, x^2, F(x^1, x^2)\}$ or $\{(x^1, F(x^1, x^3), x^3)\}$ or $\{(F(x^2, x^3), x^2, x^3)\}$.

1.2. Lie derivative.

Let M be a smooth manifold.

1. Let $D\colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ be a derivation, i.e. D is $\mathbb{R}$-linear and satisfies $D(fg) = fD(g) + gD(f)$ for all $f, g \in \mathcal{C}^\infty(M)$. Show that there exists a unique vector field $V \in \Gamma(TM)$ so that $D(f) = Vf$ for all $f \in \mathcal{C}^\infty(M)$.
2. Let $V, W \in \Gamma(TM)$. Show that $[V, W] \in \Gamma(TM)$ where $[V, W]\colon f \mapsto VWf - WVf$. Express V in a local chart (φ, U) as $V = V^i \partial_{\varphi^i}$ where $V^i \in \mathcal{C}^\infty(\varphi(U))$, similarly $W = W^j \partial_{\varphi^j}$. Is $[V, W]$ tensorial in V, i.e. $\mathcal{C}^\infty(M)$-linear in V?

1.3. Existence of Riemannian metrics.

Let M be a smooth manifold. Show that there exists a Riemannian metric g on M.
Hint. Use a partition of unity. Carefully check that your construction yields something positive definite!

1.4. Isometries.

1. Show that the map

$$\Big((0, \infty) \times (0, 2\pi), \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2\Big) \ni (r, \phi) \overset{F}{\mapsto} (r\cos\phi, r\sin\phi) \in \Big(\mathbb{R}^2, (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\Big)$$

   is a local isometry.

2. Show that Möbius transformations $z \overset{A}{\mapsto} \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, are isometries of $(\mathbb{H}^2, g)$ where we define $\mathbb{H}^2 := \{z \in \mathbb{C}\colon \Im z > 0\}$ and $g_{x+iy} = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. Show that $(\mathbb{H}^2, g)$ is homogeneous and isotropic.