10. Hadamard manifolds

10.1. Geodesics in Hadamard manifolds.

Let γ be an isometry of a Hadamard manifold (M, g). Show that $\mathrm{Min}(\gamma)$ is closed, geodesically convex (i.e. if $\gamma\colon [0,1] \to \mathrm{Min}(\gamma)$ is a geodesic with $\gamma(0), \gamma(1) \in \mathrm{Min}(\gamma)$, then also $\gamma(t) \in \mathrm{Min}(\gamma)$ for all $t \in [0,1]$), and γ-invariant.

10.2. "Uniqueness" and symmetries of hyperbolic space.

Prove that if M is a n-dimensional Riemannian manifold satisfying properties

1. for any given point all geodesic rays $x(t)$, $t \geq 0$ emanating from it are minimizing up to arbitrarily large values of $t > 0$ (note that this is stronger than geodesic completeness).
2. the sectional curvatures are constantly equal to -1,

and $p \in M$ then $\exp_p$ induces an isometry between $\mathbb{R}^n$ with metric

$$g(w, w) = \Big(w \cdot \tfrac{x}{|x|}\Big)^2 + \Big(|w|^2 - \Big(w \cdot \tfrac{x}{|x|}\Big)^2\Big)\frac{\sinh^2|x|}{|x|^2} \tag{1}$$

and M. Deduce that given any two points p, q in the hyperbolic space $\mathbb{H}$ and any isometry H between their tangent spaces $T\mathbb{H}_p \to T\mathbb{H}_q$ there is a unique isometry $f : \mathbb{H} \to \mathbb{H}$ such that $f(p) = q$ and $df_p = H$.

10.3. Two dimensional Hadamard manifolds.

Let (M, g) be a two dimensional Hadamard manifold. For fixed point $p \in M$ and isometry $H : \mathbb{R}^2 \to TM_p$, consider $(\mathbb{R}^2, \overline{g})$ where $\overline{g} := (\exp_p \circ H)^* g$.

1. Show that $\overline{g}$ is of the form

$$\overline{g}_x(v, w) := \Big(v \cdot \tfrac{x}{|x|}\Big)\Big(w \cdot \tfrac{x}{|x|}\Big) + \frac{f^2(x)}{|x|^2}\Big(v \cdot w - \Big(v \cdot \tfrac{x}{|x|}\Big)\Big(w \cdot \tfrac{x}{|x|}\Big)\Big), \tag{2}$$

   where $f^2(x)/|x|^2$ is smooth (also at $x = 0$) and has limit 1 as $x \to 0$, and where $t \mapsto f(tx)$ is nonnegative and convex for any fixed $x \in \mathbb{R}^2 \setminus \{0\}$.

2. Reciprocally, show that $\mathbb{R}^2$ endowed with any metric $\overline{g}$ satisfying the properties established in 1. —and such that $g_x(v, w)$ extends to a smooth metric across $x = 0$— gives a model of a Hadamard manifold (simply connected with nonpositive sectional curvature at all points).

10.4. Asymptotic expansion of the circumference.

Let M be a manifold, $E \subset TM_p$ a linear 2-plane and $\gamma_r \subset E$ a circle with center 0 and radius $r > 0$ sufficiently small. Show that

$$L(\exp(\gamma_r)) = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right)$$

for $r \to 0$.