12. Differential forms, part two

12.1. Application of Stokes' theorem.

Let M be a smooth oriented manifold. Write $\omega \in \Omega^m(M)$ for the volume form of a Riemannian metric g on M. Let $V \in \Gamma(TM)$.

1. Show that $d(i_V\omega) = \mathrm{d}iv_g(V)\omega$.
2. Write h for the metric on ∂M induced by g via pullback along the inclusion map $i\colon \partial M \hookrightarrow M$, and write $\sigma \in \Omega^{m-1}(\partial M)$ for the volume form of $(\partial M, h)$. Show that $\sigma = i_N\omega$ where $N \in \mathcal{C}^\infty(\partial M; T_{\partial M}M)$ is an outward pointing unit normal vector field (i.e. $g(N,N) = 1$ and $N \perp T\partial M$). Prove also the existence of such an N.
3. Prove the *divergence theorem*: $\int_M \mathrm{d}iv_g(V)\omega = \int_{\partial M} g(V,N)\sigma$.

12.2. Poincare Lemma.

The goal of this exercise is to prove the Poincaré lemma for compactly supported cohomology in the following form: for a smooth manifold M, we have $H^{k+1}_{\mathrm{c}}(\mathbb{R}\times M) \cong H^k_{\mathrm{c}}(M)$.

1. Define the map $\pi_*\colon \Omega^{k+1}(\mathbb{R}\times M) \to \Omega^k(M)$, given by *integration on the fibers*, as follows: for $\omega = \mathrm{d}t \wedge \omega_0 + \omega_1$ with $\omega_j = \sum_I \omega_{j,I}(t,x)\mathrm{d}x^I$, $j = 0,1$, where the $\omega_{j,I}$ have compact support in t, set

   $$\pi_*\omega := \int_{-\infty}^{\infty} \omega_0(t)\,\mathrm{d}t := \sum_I \left(\int_{-\infty}^{\infty} \omega_{0,I}(t,x)\,\mathrm{d}t\right)\mathrm{d}x^I.$$

   Show that $\pi_*\mathrm{d} = \mathrm{d}\pi_*$. Therefore, π_* induces a map in cohomology which we denote

   $$\pi_*\colon H^{*+1}(\mathbb{R}\times M) \to H^*(M).$$

2. Let $e = e(t)\,\mathrm{d}t$ where $e \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ has total integral 1. Define the map

   $$e_*\colon \Omega^*_{\mathrm{c}}(M) \to \Omega^{*+1}_{\mathrm{c}}(M\times\mathbb{R}), \qquad \phi \mapsto \phi\wedge e.$$

   Show that e_* induces a map in cohomology which we denote also by

   $$e_*\colon H^*(M) \to H^{*+1}(\mathbb{R}\times M).$$

3. Show that $\pi_* \circ e_* = \mathrm{I}d$ on $H^*(M)$.

4. Show that $e_* \circ \pi_* = \mathrm{I}d$ on $H^*(\mathbb{R}^\times M)$ as follows. Define the map

$$\begin{aligned} K\colon \Omega^*_{\mathrm{c}}(\mathbb{R}\times M) &\to \Omega^{*-1}_{\mathrm{c}}(\mathbb{R}\times M),\\ K(f\pi^*\phi) &:= 0 \quad (f\in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}\times M),\ \phi\in\Omega^k_{\mathrm{c}}(M)),\\ K(f\pi^*\phi\wedge \mathrm{d}t)(t) &= (\pi^*\phi)\int_{-\infty}^t f - \phi A(t)\int_{-\infty}^{\infty} f \end{aligned}$$

where $A(t)=\int_{-\infty}^t e$. Verify that

$$1-e_*\pi_* = (-1)^k(\mathrm{d}K - K\mathrm{d}) \quad \text{on} \quad \Omega^{k+1}_{\mathrm{c}}(\mathbb{R}\times M)$$

and use this to conclude the argument.

12.3. Cohomology of the sphere.

The goal of this exercise is to show that the cohomology groups of $\mathbb{S}^m$ are given by $H^k(\mathbb{S}^m)\cong\mathbb{R}$ for $k=0,m$, and 0 for all other k. Proceed by induction on m using the Mayer–Vietoris sequence. (The case $m=1$ may be assumed, as it was discussed in class.)

12.4. Metrics of negative sectional curvature.

The goal of this exercise is to prove the following theorem: Let M and N be compact, connected, smooth manifolds of positive dimension. Then $M\times N$ does not admit a metric of negative sectional curvature.

Proceed as follows:

1. Show that at least one of the factors M or N is simply connected.
2. Assume that M is simply connected. Show that the universal covering space of $M\times N$ is $M\times\widetilde{N}\cong\mathbb{R}^{m+n}$ where $\widetilde{N}$ is the universal covering space of N.
3. Show that $\mathbb{R}^{m+n}$ cannot be diffeomorphic to $M\times\widetilde{N}$ by considering an m-form $\omega\in\Omega^m(M)$ with $\int_M\omega=0$ and computing $\int_{s(M)}\pi^*\omega$, where $s\colon M\to M\times\widetilde{N}$, $s(p)=(p,q_0)$ (for some fixed $q_0\in\widetilde{N}$) and $\pi\colon M\times N\to M$ is the projection, in two different ways. (Hint: what is $\mathrm{d}\pi^*\omega$?)