2. Length and volume, Levi-Civita connection, vector fields

2.1. Parametrizations, lengths and volumes.

As discussed in the lectures, for a smooth curve $\gamma : [a,b] \to M$, into a Riemannian manifold (M,g), one defines its length by

$$L(\gamma) = \int_a^b g(\gamma'(t), \gamma'(t))^{1/2} dt.$$

We also define its energy

$$E(\gamma) = \int_a^b g(\gamma'(t), \gamma'(t)) dt.$$

1. Prove that $L(\gamma)$ is independent of the parametrization: for any diffeomorphism $\psi : [c,d] \to [a,b]$, $L(\gamma \circ \psi) = L(\gamma)$.

2. Prove that in general $E(\gamma \circ \psi) \neq E(\gamma)$. Give sufficient conditions on ψ, such that equality holds.

3. Let $\mathcal{A}$ be the family of diffeomorphisms between $[a,b]$ and $[c,d]$ for arbitrary $c \neq d$. Prove that

$$\sup_{\psi \in \mathcal{A}} E(\gamma \circ \psi) = \infty, \qquad \inf_{\psi \in \mathcal{A}} E(\gamma \circ \psi) = 0$$

4. Bonus question, to be discussed later: what happens if the length of $[c,d]$ is prescribed?

2.2. Connections.

Let M be an m-dimensional smooth manifold. Suppose for all $V, W \in \Gamma(TM)$ we are given $D_V W$ with the following properties for all $f \in \mathcal{C}^\infty(M)$, $V, W \in \Gamma(TM)$:

$$D_{fV} W = f D_V W, \qquad D_V(fW) = (Vf)W + f D_V W.$$

$$D_{V_1+V_2} W = D_{V_1} W + D_{V_2} W \quad D_V(W_1 + W_2) = D_V W_1 + D_V W_2 \,.$$

One calls D a **connection**.

1. Show that in local coordinates $x = (x^1, \ldots, x^m)$ on M there exist smooth functions Γ_{ij}^k (called **connection coefficients**, $i,j,k = 1, \ldots, m$, so that $D_V W = V^i (\partial_{x^i} W^j) \partial_{x^j} + V^i W^j \Gamma_{ij}^k \partial_{x^k}$.

2. Show, conversely, that this formula defines a map (in the local coordinate chart) satisfying the properties above.

3. Show that D is **torsion-free**, meaning $D_V W - D_W V = [V, W]$ for all $V, W \in \Gamma(TM)$, if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$.

4. Show that there exists a connection D.

5. Fix a connection D_0. Prove that $\{D - D_0 \colon D \text{ is a connection}\} \cong \Gamma(T_{1,2}M)$ via $D \mapsto \Big((V, W) \mapsto D_V W - (D_0)_V W\Big)$. Thus, the space of connections is an infinite-dimensional affine space modelled on $\Gamma(T_{1,2}M)$.

6. State (and prove) an analogous result for connections on a vector bundle $E \to M$.

2.3. Levi-Civita connection of immersed submanifold.

1. Let $(\overline{M}, \overline{g})$ be a Riemannian manifold with Levi-Civita connection $\overline{D}$, and let M be a submanifold of $\overline{M}$, equipped with the induced metric $g := i^*\overline{g}$, where $i : M \to \overline{M}$ is the inclusion map. Show that the Levi-Civita connection D of (M, g) satisfies $D_X Y = (\overline{D}_X Y)^T$ for all $X, Y \in \Gamma(TM)$, where the superscript T denotes the component tangential to M and $\overline{D}_X Y$ is <u>defined</u> as $\overline{D}_X Y := \overline{D}_{\overline{X}} \overline{Y}$ for any extensions $\overline{X}, \overline{Y} \in \Gamma(TM)$ of X, Y.

2. Let (M, g) be a smooth manifold with Levi-Civita connection D. Consider the metric $\tilde{g} = \varphi g$ for a positive smooth function $\varphi : M \to \mathbb{R}_{>0}$. Compute the Levi-Civita connection of $(M, \tilde{g})$. What happens for $\varphi \equiv c > 0$? How do you explain it?
   Hint: By problem 2.2. any two connections differ by a tensor, try to compute that tensor to obtain the result.

2.4. Pullbacks.

Let N be a smooth manifold, and let (M, g) be a Riemannian manifold. Let $F \colon N \to M$ be a smooth map.

1. Let $V \colon N \to TM$ be a vector field along F. Let $p \in N$. Show that there exist a neighborhood $U \subset N$ of p, smooth vector fields $V_1, \ldots, V_m \in \Gamma(TM)$, and smooth functions $f^1, \ldots, f^m \in \mathcal{C}^\infty(N)$ so that $V = f^i F^*(V_i)$ on U. (Here $F^*(V_i) \colon N \to TM$ is the vector field $N \ni q \mapsto V(F(q)) \in T_{F(q)}M$ along F.)

2. Let $V, W \colon N \to TM$ be vector fields along F and $Z \in \Gamma(TN)$. Write $\langle V, W\rangle(p) = g_{F(p)}(V_p, W_p)$. Show that $Z\langle V, W\rangle = \langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle$.
   Hint. Show this first in the case that V, W are pullbacks along F of smooth vector fields on M. Conclude in the general case using the first part.