3. Affine connections and geodesics

3.1. Connection along integral curves of vector fields.

Let X and Y be smooth vector fields on M, i.e. $X, Y \in \Gamma(TM)$. Let ∇ denote the Levi-Civita connection of (M, g). Let $p \in M$ and $\gamma : [0,1] \to M$ the integral curve of X through p. Recall that by definition, the integral curve γ satisfies the initial value problem

$$\begin{cases} \gamma(0) = p \\ \frac{\partial \gamma}{\partial t} = X(\gamma(t))\,. \end{cases} \tag{1}$$

Prove that ∇ is given by

$$\nabla_X Y(p) = \frac{d}{dt}(P^{-1}_{\gamma,0,t}(Y(c(t)))\bigg|_{t=0}, \tag{2}$$

where $P_{\gamma,0,t} : T_{\gamma(0)}M \to T_{\gamma(t)}M$ is the parallel transport along γ from 0 to t.
Hint: prove that you can extend a basis $e_1, ..., e_m$ of T_pM to a local frame $e_1(t), ..., e_m(t) \in T_{\gamma(t)}M$ along γ such that $e_i(t)$ is parallel. Consider the restriction of Y on $\gamma([0,1])$ expressed using the parallel frame.

3.2. Connection is determined by geodesics and torsion tensor.

Let ∇ and $\tilde{\nabla}$ be two connections on TM. We call a curve $\gamma : [0,1] \to M$ a geodesic if it is self-parallel, that is $\nabla_{\partial_t}\gamma' = 0$ (see Definition 3.18 in the lecture notes). Prove that the following two are equivalent:

1. $\nabla \equiv \tilde{\nabla}$, the two connections are identical
2. ∇ and $\tilde{\nabla}$ have the same geodesics and the same torsion fields $T_\nabla \equiv T_{\tilde{\nabla}}$.

Hint: by Definition 3.5, $T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ and $T_\nabla(X,Y) = -T_\nabla(Y,X)$. By exercise 2.5, the difference of two connections satisfies $A = \nabla - \tilde{\nabla} \in \Gamma(T_{1,2}M)$. Decompose A in its symmetric A^s and anti-symmetric A^a part and try to understand what the conditions in 2. mean for A^a and A^s respectively.

3.3. Parallel transport on the 2-sphere.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere endowed with the Levi-Civita connection on TS^2. Let $\gamma : [0,1] \to S^2$ be a smooth curve.

1. Prove that parallel transport is independent of reparametrization: if V is a parallel vector field along γ, then for any diffeomorphism between intervals (reparametrization) $\alpha : [a,b] \to [0,1]$, $V \circ \alpha$ is a vector field along $\gamma \circ \alpha$ and it is parallel. (For this part of the exercise, $M = S^2$ is not needed, and does not simplify the proof, the connection can be chosen to be arbitrary).

2. Prove that a vector field V along γ is parallel if and only if $\partial_t V(t) \in \mathbb{R}^3$ is orthogonal to the sphere. *Hint:* use a parallel frame along γ (see also the hint to Problem 3.1) and the compatibility of the metric applied to sections of constant length.

3. What is the parallel transport along geodesics on S^2? (By symmetry, it suffices to choose a geodesic in a convenient system of coordinates)

4. What is the parallel transport along curves that parametrize $S^2 \cap \{z = c\}$ for $c \in (-1,1)$?

3.4. Parallel transport and geodesics on hyperbolic plane.

Recall the hyperbolic plane introduced in exercise 1.4:

$$\mathbb{H}^2 := \{z \in \mathbb{C} \colon \Im z > 0\} \quad \text{and} \quad g_{x+iy} = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2).$$

1. Compute the Christoffel symbols of the Levi-Civita connection of $(\mathbb{H}^2, g)$.

2. Let $v = (0,1)$ be a tangent vector at $p = (0,1) \in \mathbb{H}^2$. (This means that under the canonical identification of $T_p\mathbb{H}^2$ with $\mathbb{R}^2$, the vector v_0 is identified with $(0,1)$.) Let $v(t)$ be the parallel transport of v along the curve $\gamma(t) = (t,1)$. Show that $v(t)$ makes an angle t with the direction of the y-axis.

3. Show that vertical lines are geodesics in $(\mathbb{H}^2, g)$. Show that any geodesic is either a vertical half line or a half circle intersecting the x-axis orthogonally.

Hint for 2: In the proof of 3.14 you saw the defining ODE for parallel transport in local coordinates. In this case, writing $v(t) = (w(t), z(t))$, deduce that the parallel transport system simplifies to

$$\begin{cases} \dot{w}(t) + \Gamma^1_{12}(\pi(v(t)))z = 0 \\ \dot{z}(t) + \Gamma^2_{11}(\pi(v(t)))w = 0 \end{cases} \tag{3}$$

Since parallel transport is an isometry, parametrize the curve $v(t)$ in $T\mathbb{H}^2$ using just one parameter $\theta(t)$, $v(t) = (\cos(\theta(t)), \sin(\theta(t)))$.