4. Geodesics, Hopf-Rinow theorem

4.1. Geodesic variations.

1. Let $\gamma : [0,1] \to M$ be a smooth curve in a Riemannian manifold (M, g). Let V be a vector field along γ with $V(0) = V(1) = 0$. Show that there exists

$$\tilde{\gamma} : (-1,1) \times [0,1] \to M$$

satisfying $\tilde{\gamma}(0,t) = \gamma(t)$, $\tilde{\gamma}(0,s) = \gamma(0)$, $\tilde{\gamma}(1,s) = \gamma(1)$ and such that V is the variation vector field of $\tilde{\gamma}$, i.e. $V(t) = \partial_s \tilde{\gamma}(s,t)$.

2. Let $\gamma : [0,1] \times [0,a] \to M$ be a smooth map such that for fixed $a_0 \in [0,a]$, $\gamma_{a_0} : t \mapsto \gamma(t, a_0)$ is a geodesic parametrized by arc length. Assume that the curve $b \mapsto \gamma(0,b)$ is orthogonal to the curve γ_{a_0} at the point $\gamma(0,a_0)$. Prove that for all $(t_0, a_0) \in [0,1] \times [0,a]$ the curves $b \mapsto \gamma(t_0, b)$ and γ_{a_0} are orthogonal where they intersect.

4.2. Exponential map on $SO(n)$.

Consider $M = SO(n) \subset \mathbb{R}^{n \times n}$ with the induced metric. Consider $p = I \in SO(n)$, show that

1. $T_p M = \{B \in \mathbb{R}^{n \times n} \,|\, B + B^T = 0\}$
2. $\exp_p(B) = \sum_{i=0}^{\infty} \frac{1}{i!} B^i$ (matrix exponential).

4.3. Riemannian structure on TTM.

Let $(p, v) \in TM$ and $V, W \in T_{(p,v)}TM$. Choose curves in TM with

$$\alpha : t \mapsto (p(t), v(t)) \quad \beta : s \mapsto (q(s), w(s)),$$

$$p(0) = q(0) = p \quad v(0) = w(0) = v \quad \alpha'(0) = V \quad \beta'(0) = W.$$

Define an inner product on TM by

$$g_{(p,v)}(V, W) = g_p^M(d\pi(V), d\pi(W)) + g_p^M(\nabla^M_{\partial_t} v(0), \nabla^M_{\partial_t} w(0)) \,.$$

1. Prove that this formula defines a well-defined Riemannian metric on TM.

2. A vector $(p, v) \in TM$ that is orthogonal to the vectors tangent to the fiber $\pi^{-1}(p) \cong T_pM$ is called a hotizontal vector. A curve $\gamma : t \mapsto (p(t), v(t)) \in TM$ is defined to be horizontal if its tangent vector $\gamma'(t) \in T_{\gamma(t)}TM$ is horizontal for all t.
   Prove that the curve $\gamma(t)$ is horizontal if and only if $v(t)$ is parallel along $p(t)$ with respect to the Riemannian structure and Levi-Civita connection of M.

3. Prove that the geodesic vector field on TTM (see also proof of 4.1) is horizontal at every point.

4. Prove that the flow lines of the geodesic vector field are geodesics on TM for the metric introduced in 1.

4.4. Applications of Hopf-Rinow.

Let (M, g) be a homogeneous Riemannian manifold, i.e. the isometry group of M acts transitively on M. Prove that for any two points $p, q \in M$ there exists a geodesic γ between them satisfying $L(\gamma) = d(p, q)$.

4.5. Existence of closed geodesics.

Let (M, g) be a compact Riemannian manifold and $c_0 \colon S^1 \to M$ a continuous closed curve. The purpose of this exercise is to show that in the family of all continuous and piece-wise C^1 curves $c \colon S^1 \to M$ which are homotopic to c_0, there is a shortest one and it is a geodesic.

a) Show that c_0 is homotopic to a piece-wise C^1-curve c_1 with finite length.

b) Let $L \coloneqq \inf_c L(c)$ be the infimum over all piece-wise C^1 curves $c \colon S^1 \to M$ homotopic to c_0 and consider a minimizing sequence $(c_n \colon S^1 \to M)_n$ with $\lim_n L(c_n) = L$. Use compactness of M to construct a piece-wise C^1-curve $c \colon S^1 \to M$ with length L.
Hint. Cover M with simply connected balls with the property that every two points in a ball are joined by a unique distance minimizing geodesic.

c) Conclude by showing that c is homotopic to c_0 and a geodesic.