5. Curvature

5.1. Ricci curvature.

Let (M, g) be a 3-dimensional Riemannian manifold. Show the following:

1. The Ricci curvature Ric uniquely determines the Riemannian curvature tensor R.
2. If M is an Einstein manifold, that is, a Riemannian manifold (M, g) with $\mathrm{Ric} = kg$ for some $k \in \mathbb{R}$, then the sectional curvature sec is constant.

5.2. Metric and Riemannian isometries.

Let (M, g) and $(\bar{M}, \bar{g})$ be two connected Riemannian manifolds with induced distance functions d and $\bar{d}$, respectively. Further, let $f\colon (M, d) \to (\bar{M}, \bar{d})$ be an isometry of metric spaces, i.e. f is surjective and for all $p, p' \in M$ we have $\bar{d}(f(p), f(p')) = d(p, p')$.

1. Prove that for every geodesic γ in M, $\bar{\gamma} := f \circ \gamma$ is a geodesic in N.
2. Let $p \in M$. Define $F\colon TM_p \to T\bar{M}_{f(p)}$ with
$$F(X) := \frac{d}{dt}\bigg|_{t=0} f \circ \gamma_X(t),$$
where γ_X is the geodesic with $\gamma_X(0) = p$ and $\dot{\gamma}(0) = X$. Show that F is surjective and satisfies $F(cX) = cF(X)$ for all $X \in TM_p$ and $c \in \mathbb{R}$.
3. Conclude that F is an isometry by proving $\|F(X)\| = \|X\|$.
4. Prove that F is linear and conclude that f is smooth in a neighborhood of p.
5. Prove that f is a diffeomorphism for which $f^*\bar{g} = g$ holds.

5.3. Flat manifolds.

Consider the torus $\mathbb{T}^m = S^1 \times ... \times S^1$ endowed with the product metric coming from m-times the standard metric on S^1.

1. Express the metric g in local coordinates.
2. Show that this metric on the torus $(\mathbb{T}^m, g)$ defines a <u>flat manifold</u> (a manifold for which $K(\Pi) = 0$ for every plane $\Pi \subset T_pM$ and every $p \in M$).
3. Decide whether this statement is true or false: "A smooth Riemannian manifold is flat if and only if the Riemann curvature tensor vanishes identically."

5.4. Curvatures of spheres.

Let $S_r^m \subset \mathbb{R}^{m+1}$ be the m-dimensional sphere of radius r endowed with the standard metric.

1. Compute the Riemann curvature tensor of S_r^m.
2. Compute the Ricci curvature tensor of S_r^m.
3. Compute the scalar curvature of S_r^m.